Gravity assisted solution of the mass gap problem for pureYang-Mills fields

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In 1979 Louis Witten demonstrated that stationary axially symmetric Einstein field equations and those for static axially symmetric self-dual SU(2) gauge fields can both be reduced to the same (Ernst) equation. In this paper we use this result as point of departure to prove the existence of the mass gap for quantum source-free Yang-Mills (Y-M) fields. The proof is facilitated by results of our recently published paper, JGP 59 (2009) 600-619. Since both pure gravity, the Einstein-Maxwell and pure Y-M fields are described for axially symmetric configurations by the Ernst equation classically, their quantum descriptions are likely to be interrelated. Correctness of this conjecture is successfully checked by reproducing (by different methods) results of Korotkin and Nicolai, Nucl.Phys.B475 (1996) 397-439, on dimensionally reduced quantum gravity. Consequently, numerous new results supporting the Faddeev-Skyrme (F-S) -type models are obtained. We found that the F-S-like model is best suited for description of electroweak interactions while strong interactions require extension of Witten's results to the SU(3) gauge group. Such an extension is nontrivial. It is linked with the symmetry group SU(3)×SU(2)×U(1) of the Standard Model. This result is quite rigid and should be taken into account in development of all grand unified theories. Also, the alternative (to the F-S-like) model emerges as by-product of such an extension. Both models are related to each other via known symmetry transformation. Both models possess gap in their excitation spectrum and are capable of producing knotted/linked configurations of gauge/gravity fields. In addition, the paper discusses relevance of the obtained results to heterotic strings and to scattering processes involving topology change. It ends with discussion about usefulness of this information for searches of Higgs boson.

Keywords : Extended Ricci flow; Bose-Einstein condensation; Ernst, Landau-Lifshitz, Gross-Pitaevski, Richardson-Gaudin equations; Einstein's vacuum and electrovacuum equations; Floer's theory; instantons, monopoles, calorons; knots, links and hyperbolic 3-manifolds; Standard Model; Higgs boson.

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1 Introduction

1.1 General remarks

History of physics is full of situations when experimental observations lead to deep mathematical results. Discovery of Yang-Mills (Y-M) fields in 1954 [1] falls out of this trend. Furthermore, if one believes that theory of these fields makes sense, they should never be directly observed. To make sure that these fields do exist, it is necessary to resort to all kinds of indirect methods to probe them. Physically, the rationale for the Y-M fields is explained already in the original Yang and Mills paper [1]. Mathematically, such a field is easy to understand. It is a non Abelian extension of Maxwell's theory of electromagnetism. In 1956 Utiyama [2] demonstrated that gravity, Y-M and electromagnetism can be obtained from general principle of local gauge invariance of the underlying Lagrangian. The explicit form of the Lagrangian is fixed then by assumptions about its symmetry. For instance, by requiring invariance of such a Lagrangian with respect to the Abelian U(1) group, the functional for the Maxwell field is obtained, while doing the same operations but using the Lorentz group the Einstein-Hilbert functional for gravitational field is recovered. By employing the SU(2) non Abelian gauge group the original Y-M result [1] is recovered.

Only Maxwell's electromagnetic field is reasonably well understood both at the classical and quantum level. Due to their nonlinearity, the Y-M fields are much harder to study even at the semi/classical level. In particular, no classical solutions e.g. solitons (or lumps) with finite action are known in Minkowski space-time. This result was proven by many authors, e.g. see [3-4] and refrerences therein. The situation changes dramatically in Euclidean space where the self-duality constraint allows to obtain meaningful classical solutions [5,6]. These are helpful for development of the theory of quantum Y-M fields. Such solutions are useful in the fields other than quantum chromodynamics (QCD) since the self-duality equations are believed to be at the heart of all exactly integrable systems [7]. Although the self-duality equations originate from study of the Y-M functional, not all solutions [6] of these equations are relevant to QCD. In this paper we discuss the rationale behind the selection procedure. In QCD solutions of self-duality equations, known as *instantons*, are describing tunneling between different QCD vacua [8]. It should be noted though that treatment of instantons in mathematics [9-11] and physics literature [8] is different. This fact is important. It is important since one of the major tasks of nonperturbative QCD lies in developing mathematically correct and physically meaniful description of these vacua. According to a point of view existing in physics literature the QCD has a countable infinity of topologically different vacua. Supposedly, the Faddeev-Skyrme (F-S) model is designed for description of these vacua. If this model can be used for this purpose, then

each vacuum state is expected to be associated with a particular knot (or link) configuration. Under these conditions the instantons are believed to be well localized objects interpolating between different knotted/linked vacuum configurations [12-16]. These configurations upon quantization are expected to posses a tower of excited states. Whether or not such a tower has a gap in its spectrum or the spectrum is gapless is the essence of the millennium prize problem[1]. Originally, the above results were obtained and discussed only for SU(2) gauge fields [17]. They were extended to SU(N) case, $N \geq 2$, only quite recently [18][2]. Although such a description of QCD vacua is in accord with general principles of instanton calculations [8], it is in *formal* disagreement with results known in mathematics [9-11]. Indeed, it is well known that complement of a particular knot in S^3 is 3-manifold. Since instantons "live" in $\mathbf{R}^4$ (or any Riemannian 4-manifold allowing an anti self -dual decomposition of the Y-M field (e.g. see Ref.[9], pages 38-39 [3]), this means that all knots in $\mathbf{R}^4$(or S^4) are trivial and one should talk about knotted spheres instead of knotted rings [20]. This known topological fact is in apparent contradiction with results of [13-15]. In this work we shall provide evidence that such a contradiction is only apparent and that, indeed, knotted configurations in S^3 are consistent with the notion of instantons as formulated in mathematics. This is achieved by using results by Floer [21]. It should be noted, though, that known to us "proofs" [22-24] of the existence of the mass gap in pure Y-M theory done at the physical level of rigor ignore instanton effects altogether. Among these papers only Ref.[22] uses the F-S SU(2) model for such mass gap calculations. It also should be noted that results of such calculation sensitively depend upon the way the F-S model is quantized. For instance, in the work by Faddeev and Niemi [25], done for the SU(2) gauge group, the results of quantization produce gapless spectrum. To fix the problem the same authors suggested to extend the original model in ad hock fashion. Other authors, e.g. see Ref.[26], proposed different ad hoc solution of the same problem.

The above results are formally destroyed by the effects of gravity. Indeed, in 1988 Bartnik and McKinnon numerically demonstrated [27] that the combined Y-M and gravity fields lead to a stable particle-like (solitonic) solutions while neither source-free gravity nor pure Y-M fields are capable of producing such solutions[4]. Such situation has interesting cosmological ramifications[5] [28] causing disappearance of singularities in spacetime as shown by Smoller et al [29]. In this work we do not discuss implications of these remarkable results. Instead, in the spirit of Floer's ideas [21], we argue that even without taking these results into account, the effects of gravity on processes of high energy physics are quite substantial.

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[1]E.g. see

http://www.claymath.org/millennium/Yang-Mills_Theory/

[2]In this work, in accord with experimental evidence, we demonstrate that $N \leq 3$.

[3]In physics literature, both anti and self dual instantons are allowed to exist, e.g. see Ref.[19], page 481.

[4]More accurately, neither pure Y-M fields nor pure gravity have nontrivial static globally regular (i.e.nonsingular, asymptotically flat) solitons.

[5]E.g. Einstein-Y-M hairy black holes

1.2 Statements of the problems to be solved

In this paper several problems are posed and solved. In particular, we would like to investigate the physics and mathematics behind gravity-Y-M correspondence discovered by Louis Witten [30] for SU(2) gauge fields. Is this correspondence accidental? If it is not accidental, how it should be related to commonly shared opinion that the Standard Model (SM) of particle physics does not account for gravity? Can this correspondence be extended to other gauge fields, e.g. SU(N), N>2 ? If the answer is "yes", will such correspondence be valid for all N's or just for few? In the last case, what such a restriction means physically? How the noticed correspondence is helping to solve the gap problem? What role the F-S model is playing in this solution? Is this model instrumental in solving the gap problem or are there other aspects of this problem which the F-S model is unable to account? How this correspondence affects known string-theoretic and loop quantum gravity (LQG) results ? What place the topology-changing (scattering) processes occupy in this correspondence? Is there any relevance of the results of this work to searches for Higgs boson?

1.3 Organization of the rest of the paper and summary of obtained results

Sections 2,3 and 6, and Appendix A are devoted to detailed investigation of gravity-Y-M correspondence. Section 4 is devoted to the physics-style exposition of works by Andreas Floer [11, 21] on Y-M theory with purpose of connecting his mathematical formalism for Y-M fields with the F-S model. In the same section we also consider the Y-M fields monopole and instanton solutions and their meaning and place within Floer's theory. Our exposition is based on results of Sections 2 and 3. Section 5 is entirely devoted to solution of the gap problem for pure Y-M fields. Although the solution depends on results of previous sections, numerous additional facts from statistical mechanics and nuclear physics are being used. In Section 6 we discuss various implications/corollaries of the obtained results, especially for the SM of particle physics. In Section 7 we discuss possible directions for further research based on the results presented in this paper. These include (but not limited to): connections with the LQG, the role and place of the Higgs boson, relationship between real space-time scattering processes of high energy physics and processes of topology change associated with such scattering. Based on the results of this paper, we argue that this task can be accomplished with help of the formalism developed by G. Perelman for his proof of the Poincare′ and geometrization conjectures.

The major new results of this paper are summarized as follows.

1. In subsection 5.4.4, while solving the gap problem, we reproduced by employing entirely different methods, the main results of the paper by Korotkin and Nicolai [31] on quantizing dimensionally reduced gravity. From these results it follows that for gravity and Y-M fields possessing the same symmetry the nonperrturbative quantization proceeds essentially in the same way.

2. In subsection 6.3 we demonstrated that gravity-Y-M correspondence dis-

covered by L.Witten for gauge group SU(2) can be extended *only* to the SU(3) gauge group. This group contains SU(2)×U(1) group as a subgroup. This fact allowed us to come up with the anticipated (but never proven!) conclusion about symmetry of the SM. It is given by SU(3)×SU(2)×U(1). The obtained result is very rigid. It is deeply rooted into not widely known/appreciated (discussed in Appendix A) properties of the gravitational field. It is these properties which ultimately determine the conditions of gravity-Y-M correspondence.

3. The latest papers Refs.[32-34] are aimed at reproduction of the classification scheme of particles and fields in the SM within the framework of LQG formalism. These results match perfectly with the results of our paper because of the noticed and developed gravity-Y-M correspondence. In view of this correspondence, the results of Refs.[32-34] can be reproduced with help of *minimal* gravity model described in subsections 3.2, 3.4, and 7.2 . This minimal model has differential-geometric /topological meaning in terms of the dynamics of the extended Ricci flow [35,36]. Such a flow is the minimal extension of the Ricci flow now famous because of its relevance in proving the Poincare′ and geometrization conjectures.

4. The formalism developed in this paper explains why using pure gravity one can talk about the particle/field content of the SM. Not only it is compatible with just mentioned LQG results but also with those, coming from noncommutative geometry [37], where it is demonstrated that use of pure gravity (that is "minimal model") combined with 0- dimensional internal space is sufficient for description of the SM.

2 Emergence of the Ernst equation in pure gravity and Y-M fields

2.1 Some facts about the Ernst equation

Study of static vacuum Einstein fields was initiated by Weyl in 1917. Considerable progress made in later years is documented in Ref.[38]. To develop formalism of this paper we need to discuss some facts about these static fields. Following Wald [39], a spacetime is considered to be *stationary* if there is a one-parameter group of isometries σ_t whose orbits are time-like curves e.g. see [40]. With such group of isometries is associated a time-like Killing vector ξ^i. Furthermore, a spacetime is *axisymmetric* if there exists a one-parameter group of isometries χ_ϕ whose orbits are closed spacelike curves. Thus, a spacelike Killing vector field ψ^i has integral curves which are closed. The spacetime is *stationary and axisymmetric* if it possesses both of these symmetries, provided that $\sigma_t \circ \chi_\phi = \chi_\phi \circ \sigma_t$. If $\xi = (\frac{\partial}{\partial t})$ and $\psi = (\frac{\partial}{\partial \phi})$ so that $[\xi, \psi] = 0$, one can choose coordinates as follows: $x^0 = t, x^1 = \phi, x^2 = \rho, x^3 = z$. Under such identification, the metric tensor $g_{\mu\nu}$ becomes a function of only x^2 and x^3. Explicitly,

$$ds^2 = -V(dt - wd\phi)^2 + V^{-1}[\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + dz^2)], \qquad (2.1)$$

where functions V, w and γ depend on ρ and z only. In the case when $V = 1, w = \gamma = 0$, the metric can be presented as $ds^2 = -(dt)^2 + (d\tilde{s})^2$, where

$$(d\tilde{s})^2 = \rho^2 d\phi^2 + d\rho^2 + dz^2 \qquad (2.2)$$

is the standard flat 3 dimensional metric written in cylindrical coordinates. The four-dimensional set of vacuum Einstein equations $R_{ij} = 0$ with help of metric given by Eq.(2.2) acquires the following form

$$\boldsymbol{\nabla} \cdot \{V^{-1}\boldsymbol{\nabla}V + \rho^{-2}V^2 w \boldsymbol{\nabla}w\} = 0 \qquad (2.3a)$$

and

$$\boldsymbol{\nabla} \cdot \{\rho^{-2}V^2\boldsymbol{\nabla}w\} = 0. \qquad (2.3b)$$

In these equations $\boldsymbol{\nabla}\cdot$ and $\boldsymbol{\nabla}$ are three-dimensional flat (that is with metric given by Eq.(2.2)) divergence and gradient operators respectively. In addition to these two equations, there are another two needed for determination of factor γ in the metric, Eq.(2.1). They require knowledge of V and w as an input. Solutions of Eq.s(2.3) is described in great detail in the paper by Reina and Trevers [41] with final result:

$$(\mathrm{Re}\,\epsilon)\,\nabla^2\epsilon = \boldsymbol{\nabla}\epsilon \cdot \boldsymbol{\nabla}\epsilon. \qquad (2.4)$$

This equation is known in literature as the Ernst equation. The complex potential ϵ is defined in by $\epsilon = V + i\omega$ with V defined as above and ω being an auxiliary potential whose explicit form we do not need in this work. As it was recognized by Ernst [42,43] such an equation can be also used for description of the combined Einstein-Maxwell fields. We shall exploit this fact in Section 6. *In Appendix A and in Section 6 we provide proofs that knowledge of static vacuum solutions* of the Ernst equation *is necessary and sufficient* for restoration of static Einstein-Maxwell fields.[6] Fields other than Y-M should be also restorable [7]. To proceed, we need to list several properties of the Ernst equation to be used below. First, following [41] and using prolate spheroidal coordinates, the Ernst equation reproduces the Schwarzschild metric, and with another choice of coordinates it reproduces the Kerr and Taub-NUT metric. Thus, the Ernst equation is the most general equation describing physically interesting vacuum spacetimes compatible with the Cauchy formulation of general relativity [39,40,44,45]. Such a formulation is convenient staring point for quantization of gravitational field via superspace formalism [39] leading to the Wheeler- De Witt equation, etc. *Since in this work we advocate different approach to quantization of gravity, this topic is not being discussed further.* Second, following Ref.[38], page 283, a *stationary* solution of Einstein's field equations is called *static* if the timelike Killing vector is orthogonal to the Cauchy surface. In such a case from the Table 18.1. of the same reference it follows that the Ernst

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[6]Surprisingly, upon changes of variables in these static solutions, the exact results for propagating gravitational waves can be obtained as well.

[7]This is so because each of these fields is a source of gravitational field which, in turn, can be eliminated locally. See Appendix A.

potential ϵ is real. This observation allows us to simplify Eq.(2.4) considerably. For the sake of notational comparison with Ref.[38] we redefine the potential $\epsilon = V + i\Phi$. In the static case we have $\epsilon \equiv -F \equiv -e^{2u}$[8]. Using this result in Eq.(2.4) produces

$$\Delta_{\rho,z} u = 0, \tag{2.5}$$

where $\Delta_{\rho,z}$ is flat Laplacian written in cylindrical coordinates defined by the metric, Eq.(2.2).

2.2   Isomorphism between the SU(2) self-dual gauge and vacuum Einstein field equations

This isomorphism was discovered by Louis Witten in 1979 [30]. His work was inspired by earlier works of Ernst [42] and Yang [46]. To our knowledge, since time when Ref.[30] was published such an isomorphism was left undeveloped. In this paper we correct this omission in order to demonstrate that when both fields are mathematically indistinguishable, their quantization should proceed in the same way. The result analogous to that discovered by Witten was obtained using different arguments a year later by Forgacs, Horvath and Palla [47] and, in a simpler form, by Singleton [48]. These authors used essentially the paper by Manton, Ref.[49], in which it was cleverly demonstrated that the 't Hooft-Polyakov monopole can be obtained *without* actual use of the auxiliary Higgs field. Both Refs.[47,48] and the original paper by Witten [30] use the axial symmetry of either gravitational or Y-M fields essentially. Only in this case it can be shown that the axisymmetric version of the self-duality equations obtained by Manton can be rewritten in the form of the Ernst equation. In the light of above information, following Ref.[5]**,** we shall discuss briefly contributions of Yang and Witten. For this purpose, we need to consider first the following auxiliary system of *linear* equations

$$\mathbf{\Psi}_x = \mathbf{X}\mathbf{\Psi}; \mathbf{\Psi}_t = \mathbf{T}\mathbf{\Psi}. \tag{2.6}$$

Here $\mathbf{\Psi}_x = \frac{\partial}{\partial x}\mathbf{\Psi}$ and $\mathbf{\Psi}_t = \frac{\partial}{\partial t}\mathbf{\Psi}$. In this system $\mathbf{X}$ and $\mathbf{T}$ are square matrices of the same dimension and such that

$$\mathbf{X}_t - \mathbf{T}_x + [\mathbf{X}, \mathbf{T}] = 0 \tag{2.7}$$

This result easily follows from the compatibility condition: $\mathbf{\Psi}_{xt} = \mathbf{\Psi}_{tx}$. The matrices $\mathbf{X}$ and $\mathbf{T}$ can be realized as

$$\mathbf{X} = \begin{pmatrix} -i\zeta & q(x,t) \\ r(x,t) & i\zeta \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2.8}$$

with ζ being a spectral parameter and, A, B and C being some Laurent polynomials in ζ. The above system can be extended to four variables x_1, x_2, t_1, t_2

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[8]The minus sign in front of F is written in accord with conventions of Chapter 30.2 of the 1st edition of Ref.[38].

in a simple minded fashion as follows

$$\left(\frac{\partial}{\partial x_1} + \zeta\frac{\partial}{\partial x_2}\right)\boldsymbol{\Psi} = (\mathbf{X}_1 + i\mathbf{X}_2)\boldsymbol{\Psi}, \tag{2.9a}$$

$$\left(\frac{\partial}{\partial t_1} + \zeta\frac{\partial}{\partial t_2}\right)\boldsymbol{\Psi} = (\mathbf{T}_1 + i\mathbf{T}_2)\boldsymbol{\Psi}. \tag{2.9b}$$

In the most general case, the matrices $\mathbf{X}_1, \mathbf{X}_2, \mathbf{T}_1, \mathbf{T}_2$ are made of functions which "live" in $\mathbf{C}^4$. They are representatives of the Lie algebra $sl(n, \mathbf{C})$ of $n \times n$ trace-free matrices. The compatibility conditions for this case are equivalent to the self-duality condition for the Y-M fields associated with algebra $sl(n, \mathbf{C})$.It is instructive to illustrate these general statements explicitly.

In $\mathbf{R}^4$ the (anti)self-duality condition for the Y-M curvature reads: $*F = (-1)F$ so that for the self-dual case we obtain:

$$F_{01} = F_{23}, F_{02} = F_{31}, F_{03} = F_{12}. \tag{2.10}$$

In the "light cone" coordinates $\sigma = \frac{1}{\sqrt{2}}(x_1 + ix_2), \tau = \frac{1}{\sqrt{2}}(x_0 + ix_3)$ the Y-M field one-form can be written as $A_\mu dx^\mu = A_\sigma d\sigma + A_\tau d\tau + A_{\bar{\sigma}} d\bar{\sigma} + A_{\bar{\tau}} d\bar{\tau}$ with the overbar labeling the complex conjugation. In such notations $A_0 = \frac{1}{\sqrt{2}}(A_\tau + A_{\bar{\tau}})$, $A_1 = \frac{1}{\sqrt{2}}(A_\sigma + A_{\bar{\sigma}})$, $A_2 = \frac{1}{\sqrt{2}}(A_\sigma - A_{\bar{\sigma}})$, $A_3 = \frac{1}{\sqrt{2}}(A_\tau - A_{\bar{\tau}})$. In these notations Eq.s (2.9) acquire the following form

$$F_{\sigma\tau} = 0, \ F_{\bar{\sigma}\bar{\tau}} = 0 \text{ and } F_{\sigma\bar{\sigma}} + F_{\tau\bar{\tau}} = 0. \tag{2.11}$$

They can be obtained as compatibility condition for the isospectral linear problem

$$(\partial_\sigma + \zeta\partial_{\bar{\tau}})\boldsymbol{\Psi} = (A_\sigma + \zeta A_{\bar{\tau}})\boldsymbol{\Psi} \text{ and } (\partial_\tau - \zeta\partial_{\bar{\sigma}})\boldsymbol{\Psi} = (A_{\boldsymbol{\tau}} - \boldsymbol{\zeta} A_{\boldsymbol{\bar{\sigma}}})\boldsymbol{\Psi}, \tag{2.12}$$

where the spectral parameter is ζ and $\boldsymbol{\Psi}$ is the local section of the Y-M fiber bundle. The compatibility condition reads: $(\partial_\sigma - \zeta\partial_{\bar{\tau}})(\partial_\sigma + \zeta\partial_{\bar{\tau}})\boldsymbol{\Psi} = (\partial_\sigma + \zeta\partial_{\bar{\tau}})(\partial_\sigma - \zeta\partial_{\bar{\tau}})\boldsymbol{\Psi}$, thus leading to

$$[F_{\sigma\tau} - \zeta(F_{\sigma\bar{\sigma}} + F_{\tau\bar{\tau}}) + \zeta^2 F_{\bar{\sigma}\bar{\tau}}]\boldsymbol{\Psi} = 0. \tag{2.13}$$

This equation allows us to recover Eq.s(2.11). The first two equations of Eq.s(2.11) can be used in order to represent the A -fields as follows: $A_\sigma = (\partial_\sigma C)\,C^{-1}$, $A_{\boldsymbol{\tau}} = (\partial_\tau C)\,C^{-1}$, $A_{\bar{\sigma}} = (\partial_{\bar{\sigma}}D)\,D^{-1}$ and $A_{\bar{\tau}} = (\partial_\tau D)\,D^{-1}$, where both C and D are some matrices in the Lie group G, e.g. $G = SU(2)$. By introducing the matrix $M = C^{-1}D \in G$ the last of equations in Eq.(2.11) becomes

$$\partial_{\bar{\sigma}}(M^{-1}\partial_\sigma M) + \partial_{\bar{\tau}}(M^{-1}\partial_\tau M) = 0. \tag{2.14a}$$

Thus, the self-duality conditions for the Y-M fields are equivalent to Eq.(2.14a). For the future use, following Yang [46], we notice that in such formalism the gauge transformations for Y-M fields are expressible through $D \rightarrow DE$ and

$C \rightarrow CE$ so that $F_{\sigma\bar{\sigma}} \rightarrow E^{-1}F_{\sigma\bar{\sigma}}E$ and $F_{\tau\bar{\tau}} \rightarrow E^{-1}F_{\tau\bar{\tau}}E$ with the matrix $E = E(\sigma, \bar{\sigma}, \tau, \bar{\tau}) \in SU(2)$ leaving self-duality Eq.s(2.10) (or (2.13)) unchanged.

To connect Eq.(2.14a) with the Ernst equation, following L.Witten [30] it is sufficient to assume that the matrix M is a function of $\rho = \sqrt{x_1^2 + x_2^2}$ and $z = x_3$. In such a case it is useful to remember that $\rho^2 = 2\sigma\bar{\sigma}$ and $z = \frac{i}{\sqrt{2}}(\tau - \bar{\tau})$. With help of these facts Eq.(2.14a) can be rewritten as

$$\partial_\rho(\rho M^{-1}\partial_\rho M) + \rho\partial_z(M^{-1}\partial_z M) = 0. \qquad (2.14b)$$

By assuming that the matrix M is representable by the $SL(2, R)$-type matrix, and writing it in the form

$$M = \frac{1}{V}\begin{pmatrix} 1 & \Phi \\ \Phi & \Phi^2 + V^2 \end{pmatrix}, \qquad (2.15)$$

Eq.(2.14b) is reduced to the pair of equations

$$V\nabla^2 V = \nabla V \cdot \nabla V - \nabla\Phi \cdot \nabla\Phi \text{ and } V\nabla^2\Phi = 2\nabla V \cdot \nabla\Phi.$$

With help of the Ernst potential $\epsilon = V + i\Phi$ these two equations can be brought to the canonical form of the Ernst equation, Eq.(2.4). Below, we shall provide sufficient evidence that such a reduction of the Ernst equation is compatible with analogous reduction in instanton/monopole calculations for the Y-M fields.

3 From analysis to synthesis

3.1 General remarks

The results of previous section demonstrate that for axially symmetric fields both pure gravity and pure self-dual Y-M fields are described by the same (Ernst) equation. In this section we reformulate these results in terms of the nonlinear sigma model with purpose of using such a reformulation later in the text. To do so we need to recall some results from our recent works, Ref.s[50,51]. In particular, we notice that under conformal transformations $\hat{g} = e^{2u}g$ in d-dimensions the curvature scalar $R(g)$ changes as follows:

$$\hat{R}(\hat{g}) = e^{-2u}\{R(g) - 2(d-1)\Delta_g u - (d-1)(d-2)|\nabla_g u|^2\}. \qquad (3,1)$$

Since this equation is Eq.(2.11) of our Ref.[50] we shall be interested only in transformations for which $\hat{R}(\hat{g})$ is a constant. This is possible only if the total volume of the system is conserved. Under this constraint we need to consider Eq.(3.1) for $d = 3$ in more detail. Without loss of generality we can assume that initially $R(g) = 0$. For this case we shall write $g = g_0$ so that Eq.(3.1) acquires the form

$$\hat{R}(\hat{g}) = -2e^{-2u}[2\Delta_{g_0}u + |\nabla_{g_0}u|^2] \qquad (3.2)$$

in which Δ_{g_0} is the flat space Laplacian. Now we can formally identify it with that in Eq.(2.5). Accordingly, we shall be interested in such conformal

transformations for which $\Delta_{g_0} u = 0$ in Eq.(3.2). If they exist, Eq.(3.2) can be rewritten as

$$e^{2u}\hat{R}(\hat{g}) = -2\left(\vec{\nabla}_{g_0}u\right)\cdot\left(\vec{\nabla}_{g_0}u\right).\tag{3.3}$$

This allows us to interpret Eq.(3.3) and

$$\Delta_{g_0} u = 0 \tag{3.4}$$

as interdependent equations: solutions of Eq.(3.4) determine the scalar curvature $\hat{R}(\hat{g})$ in Eq.(3.3). Clearly, under conditions at which these results are obtained only those solutions of Eq.(3.4) should be used which yield the constant scalar curvature $\hat{R}(\hat{g})$. Eq.(3.3) contains information about the Ricci tensor. To recover this information we notice that $\hat{g}_{ij} = -e^{2u}\delta_{ij}$. Therefore we obtain:

$$\hat{R}_{ij}(\hat{g}) = 2\nabla_i u \nabla_j u, \tag{3.5}$$

in accord with Eq.(18.55) of Ref.[38] where this result was obtained by employing entirely different arguments. From the same reference we find that Eq.(3.4) comes as result of use of the contracted Bianci identities applied to $\hat{R}_{ij}(\hat{g})$[9]. It is instructive to place the obtained results into broader context. This is accomplished in the next subsection.

3.2 Connection with the nonlinear sigma model

Some time ago Neugebauer and Kramer (N-K), Ref.[38], obtained Eq.s(3.4) and (3.5) using variational principle. In less general form this principle was used previously by Ernst [42] resulting in now famous Ernst equation. Neugebauer and Kramer proposed the Lagrangian and the associated with it action functional S_{N-K} producing upon minimization both Eq.s(3.4) and (3.5). To describe these results, we also use some results by Gal'tsov [52].

The functional S_{N-K} is given by

$$\mathcal{S}_{N-K} = \frac{1}{2}\int_M \sqrt{\hat{g}}[\hat{R}(\hat{g}) - \hat{g}^{ij}G_{AB}(\boldsymbol{\varphi})\partial_i\varphi^A\partial_j\varphi^B]d^3x,\tag{3.6}$$

easily recognizable as three-dimensional nonlinear sigma model coupled to 3-d Euclidean gravity. The number of components for the auxiliary field $\boldsymbol{\varphi}$ as well as the metric tensor $G_{AB}(\boldsymbol{\varphi})$ of the target space is determined by the problem in question. In our case upon variation of S_{N-K} with respect to φ_i^A and $\hat{g}_{ij}$ we should be able to re obtain Eq.s(3.4) and (3.5). To do so, following Ref.[53], we introduce the current

$$J_i = M^{-1}\partial_i M.\tag{3.7}$$

In view of results of subsection 2.2, we have to identify the matrix M with that defined by Eq.(2.15) and, taking into account Eq.(2.14a), the index i should

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[9]Ref.[38], page 283, bottom

take two values: σ and τ. With such definitions we can replace the functional $\mathcal{S}_{N-K}$ by

$$\mathcal{S} = \frac{1}{2} \int\limits_{M} \sqrt{\hat{g}}[\hat{R}(\hat{g}) - \hat{g}^{ik}\frac{1}{4}tr(J_i \ J_k)]d^3x. \qquad (3.8)$$

The actual calculations with such type of functionals can be made using results of Ref.[53]. Thus, using this reference we obtain,

$$\hat{R}_{ij}(\hat{g}) = \frac{1}{4}tr(J_i \ J_j) \qquad (3.9)$$

and

$$\partial_i J_i \ = 0. \qquad (3.10)$$

Evidently, by construction Eq.(3.10) coincides with Eq.(2.14a) and, ultimately, with Eq.(3.4). It is also easy also to check that Eq.(3.9) does coincide with Eq.(3.5). For this purpose it is sufficient to notice that

$$tr(J_i \ J_j) = -tr(\partial_i M \partial_j M^{-1}). \qquad (3.11)$$

To check correctness of our calculations the entries of the matrix M, Eq.(2.15), can be restricted to V (that is we can put $\Phi = 0$). Since $V = -F \equiv -e^{2u}$ (e.g. see discussion prior to Eq.(2.5)), a simple calculation indeed brings Eq.(3.9) back to Eq.(3.5) as required.

It is interesting and important to observe at this point that *the equation of motion, Eq.(3.10), formally is not affected by effects of gravity.* This conclusion requires some explanation. From subsection 2.2, especially from Eq.s(2.14),(2.15), it should be clear that Eq.(3.10) is the Ernst equation determining gravitational field. Hence, it is physically wrong to expect that it is going to be affected by the effects of gravity. Eq.s(3.9) and (3.10) are the same as Eq.s(3.5) and (3.4) whose meaning was explained in the previous subsection. Clearly, the functional, Eq.(3.8), can be used for coupling of *other* fields to gravity. This is indeed demonstrated in Ref.[52]. This is done with purpose of connecting results for the nonlinear sigma models with those for heterotic strings. We would like to discuss this connection now since it will be used later in the text.

3.3   Connection with heterotic string models

The functional $\mathcal{S}$, Eq.(3.8), is related to that for the heterotic string model, e.g. see Ref.[54]. For such a model the sigma model-like functional is obtainable from 10 dimensional supersymmetric string model by means of compactification scheme (ideologically similar to that used in the Kaluza-Klein theory of gravity and electromagnetism) aimed at bringing the effective dimensionality to physically acceptable values (e.g. 2, 3 or 4). For dimensionality $D < 10$ such

compactified/reduced action functional reads (e.g. see Ref.[54], Eq.(9.1.8)):

$$S_D^{heterotic} = \int d^D x \sqrt{-\det G} e^{-2\phi} [R + 4\partial^\mu \phi \partial_\mu \phi - \frac{1}{12} \hat{H}^{\mu\nu\rho} \hat{H}_{\mu\nu\rho}$$
$$-\frac{1}{4}(M^{-1})_{ij} F_{\mu\nu}^i F^{j\mu\nu} + \frac{1}{8} tr(\partial_\mu M \partial^\mu M^{-1})]. \qquad (3.12)$$

The compactification procedure is by no means unique. There are many ways to make a compactified action to look exactly like that given by Eq.(3.8) (e.g. see [55]). Evidently, there should be a way to relate such actions to each other since they all are having the same origin - 10 dimensional heterotic string action. Because of this, we would like to make some comments on action given by Eq.(3.12) by specializing to $D = 3$ for reasons explained in Refs[**68,69**] and to be clarified below, in Section 6. Under such conditions if we require the dilaton ϕ, the antisymmetric H-field (associated with string orientation) and the electromagnetic field F to vanish, the remaining action will coincide with that given by Eq.(3.8). Because of this, the following steps can be made.

First, as explained in our work, Ref.[50], for closed 3-manifolds we can/should drop the dilaton field ϕ. Second, by properly selecting string model we can ignore the antisymmetric field H. Third, by taking into account results of Appendix A we can also drop the electromagnetic field since it can be always restored from pure gravity. Thus, we end up with the action functional $\mathcal{S}$, Eq.(3.8), which we shall call "*minimal*". In Section 6 we shall provide evidence that its minimality is deeply rooted into gravity-Y-M correspondence which does not leave much room for "improvements" abundant in physics literature. We shall begin explaining this fact immediately below and will end our arguments in Section 6.

3.4 The extended Ricci flow

Thus far use of the variational principle apparently had not brought us any new results (at least at the classical level). Situation changes in the light of recent work by List [35]. Following Ref.s[35,36], it is convenient to introduce Perelman-like entropy functional $\mathcal{F}(\hat{g}_{ij}, u, f)$

$$\mathcal{F}(\hat{g}_{ij}, u, f) = \int_M (\hat{R}(\hat{g}) - 2\left|\nabla_{\hat{g}} u\right|^2 + \left|\nabla_{\hat{g}} f\right|^2) e^{-f} dv \qquad (3.13)$$

coinciding with Eq.(7.22b) of our work, Ref.[50], when $u = 0$.[10] Because of this observation, if formally we make a replacement $\mathcal{R}(\hat{g}; u) = \hat{R}(\hat{g}) - 2\left|\nabla u\right|^2$ in Eq.(3.13), we are able to identify Eq.(3.13) with Perelman's entropy functional enabling us to follow the same steps as were made in Perelman's papers aimed at proof of the geometrization and Poincaré' conjectures. Such a program

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[10]It should be noted that there is an obvious typographical error in Eq.(7.22b): the term $\left|\nabla_h f\right|^2$ is typed as $\left|\nabla_h f\right|$.

was indeed completed in the PhD thesis by List [36]. Minimization of entropy functional $\mathcal{F}(\hat{g}_{ij}, f)$ produces the following set of equations

$$\frac{\partial}{\partial t} g_{ij} = -2(\hat{R}_{ij} + \nabla_i \nabla_j f) + 4\nabla_i u \nabla_j u, \qquad (3.14a)$$

$$\frac{\partial}{\partial t} u = \Delta_{\hat{g}} u - (\nabla u) \cdot (\nabla f), \qquad (3.14b)$$

and

$$\frac{\partial}{\partial t} f = -\hat{R} - \Delta_{\hat{g}} f + 2 |\nabla u|^2, \qquad (3.14c)$$

coinciding with Eq.s(7.28a), (7.28b) of our work, Ref.[50], when $u = 0$. In these equations $|\nabla_{\hat{g}} u|^2 = \hat{g}^{ij} \nabla_i u \nabla_j u$, etc. From the next section and results below it follows that physically we should be interested in closed 3 manifolds. For such manifolds one can use Lemma 2.13, proven by List [36], which can be formulated as follows:

Let $\hat{g}, u, f$ be a solution of Eq.s(3.14) for $t \in [0, T]$ on a closed manifold M. Then the evolution of the entropy is given by

$$\partial_t \mathcal{F}(\hat{g}_{ij}, u, f) = 2 \int_M [|\mathcal{R}_{ij}(\hat{g}; u) + \nabla_i \nabla_j f|^2 + 2(\Delta_{\hat{g}} u - (\nabla u) \cdot (\nabla f))^2] e^{-f} dv \geq 0.$$

$$(3.15)$$

Thus, the entropy is non decreasing with equality taking place if and only if the solution of Eq.(3.14) is a gradient soliton. This happens when the following two conditions hold

$$\mathcal{R}_{ij}(\hat{g}; u) + \nabla_i \nabla_j f = 0 \text{ and } \Delta_{\hat{g}} u - (\nabla u) \cdot (\nabla f) = 0. \qquad (3.16)$$

For $u = 0$ the result of Perelman, Eq.(7.30) of Ref [50], for steady gradient soliton is reobtained, as required. Since for closed compact manifolds $f = const$ Eq.s(3.16) coincide with Eq.s(3.4) and (3.5) as anticipated. *Thus, existence of steady gradient solitons in the present context is equivalent to existence of solutions of static Einstein's equations for pure gravity.* This fact alone could be mathematically interesting but requires some reinforcement to be of interest physically. We initiate this reinforcement process in the following subsection.

3.5 Relationship between the F-S and the Ernst functionals

The F-S functional was mentioned in the Itroduction. In this subsection we would like to initiate study of its connection with the Ernst functional. We begin with the following observation. In steps leading to Eq.(2.14b) (or (3.10)) the Euclidean time coordinate x_0 was eventually dropped implying that solutions of selfduality for Y-M equations, when substituted back into Y-M action functional, will produce physically meaningless (divergent) results. While in subsection 4.4 we discuss a variety of means for removing of such apparent

divergence, in this subsection we notice that already Ernst [42] suggested the action functional whose minimization produces the Ernst equation. He gave two equivalent forms for such a functional, now bearing his name. These are either

$$S_{E_1}[\epsilon] = \int_M dv \frac{\boldsymbol{\nabla}\epsilon \cdot \boldsymbol{\nabla}\epsilon^*}{(\operatorname{Re}\epsilon)^2} \tag{3.17}$$

or

$$S_{E_2}[\xi] = \int_M dv \frac{\boldsymbol{\nabla}\xi \cdot \boldsymbol{\nabla}\xi^*}{(\xi\xi^* - 1)^2}. \tag{3.18}$$

Minimization of $S_{E_1}[\epsilon]$ leads to Eq.(2.4) while functional, Eq.(3.18), is obtained from $S_{E_1}[\epsilon]$ by means of substitution: $\epsilon = (\xi - 1)/(\xi + 1)$. In both functionals dv is 3-dimensional Euclidean volume element so that apparently the manifold M is just E^3 (or, with one point compactification, it is S^3).Evidently,both $S_{E_1}[\epsilon]$ and $S_{E_2}[\xi]$ are functionals for the nonlinear sigma model. If we drop the curvature term in Eq.(3.6) such truncated functional can be identified, for example, with $S_{E_2}[\xi]$. This explains why Eq.(3.10) is formally unaffected by gravity. In mathematics literature the nonlinear sigma models are known as harmonic maps. Since Reina [56] demonstrated that the functional $S_{E_2}[\xi]$ describes the harmonic map from S^3 to $\mathbf{H}^2$, it is not too difficult to write analogous functional $S_{E_3}[\xi]$ describing the mapping from S^3 to S^2. It is given by

$$S_{E_3}[\xi] = \int_M dv \frac{\boldsymbol{\nabla}\xi \cdot \boldsymbol{\nabla}\xi^*}{(\xi\xi^* + 1)^2} \tag{3.19}$$

and is part of the F-S model. If needed, both $S_{E_2}[\xi]$ and $S_{E_3}[\xi]$ can be supplemented by additional (topological) terms which in the simplest case are winding numbers. Thus, we shall be dealing either with the truncated F-S model, Eq.(3.19), or with its hyperbolic analog, Eq.(3.18). The choice between these models is nontrivial and will discussed in detail in Section 6. To facilitate this discussion, we need to observe the following. In the static case, we argued, e.g. see Eq.(2.5), that $\epsilon = -F = -e^{2u}$. Substitution of this result back into $S_{E_1}[\epsilon]$ produces (up to a constant) the following result:

$$\tilde{S}_{E_1}[\epsilon] = \int_M dv \boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}u \tag{3.20}$$

leading to Eq.(2.5) as anticipated. At the same time, consider the following H-E action functional

$$S_{H-E}[\hat{g}] = \int_M dv \sqrt{\hat{g}}\hat{R}(\hat{g}), \tag{3.21}$$

and take into account Eq.(3.3) and the fact that $\hat{g}_{ij} = -e^{2u}\delta_{ij}$. Straightforward calculation leads us then to the result (up to a constant):

$$S_{H-E}[\hat{g}] = -\int_M dv \boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}u. \tag{3.22}$$

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The minus sign in front of the integral is important and will be explained below. Before doing so, we notice that the Ernst functional (in whatever form) is essentially equivalent to the H-E functional! Since in the original paper by Ernst M is E^3 (or S^3), apparently, such a functional should be zero. This is surely not the case in general but the explanation is nontrivial. Suppose that minimization of the Ernst functional leads to some knotted/linked structures[11]. If such knots/links are hyperbolic then, by construction, complements of these knots/links in S^3 are $\mathbf{H}^3$ modulo some discrete group. This conclusion is in accord with properties of the Ernst equation discovered by Reina and Trevers [41]. Following this reference, we introduce the complex space $\mathbf{C} \times \mathbf{C} = \mathbf{C}^2$ so that $\forall \ \mathbf{z} = (u,v) \in \mathbf{C}^2$ the scalar product $z_\alpha^* z^\alpha$ can be made with the metric $\pi_{\alpha\beta} = diag\{1, -1\}$. Furthermore, the Ernst Eq.(2.4) can be rewritten with help of substitution $\epsilon = (u - v)/(u + v)$ as the set of two equations

$$z^\alpha z_\alpha^* \nabla^2 z^\beta = 2 z_\alpha^* \boldsymbol{\nabla} z^\alpha \cdot \boldsymbol{\nabla} z^\beta. \tag{3.23}$$

Such a system of equations is invariant with respect to transformations from unimodular group SU(1,1) which is equivalent to SL(2,$\mathbf{C}$). But SL(2,$\mathbf{C}$) is the group of isometries of hyperbolic space $\mathbf{H}^3$ as was discussed extensively in our work, Ref.[57]. Thus, minimization of both the F-S and Ernst functionals should account for knotted/linked structures. This conclusion is strengthened in the next subsection.

3.6 Relationship between the Ernst and Chern-Simons functionals

Even though we need to find this relationship anticipating results of the next section, by doing so, some unexpected connections with previous subsection are also going to be revealed. For this purpose, we notice that for $u = 0$ the functional $\mathcal{F}(\hat{g}_{ij}, u, f)$ introduced earlier is just Perelman's entropy functional. As such, it was discussed in our work, Ref.[50]. Evidently, both $\mathcal{F}$ and Perelman's functional can be used for study of topology of 3-manifolds. We believe, though, that use of Perelman's functional is more advantageous as we would like to explain now. For this purpose, it is convenient to introduce the Raleigh quotient λ_g via

$$\lambda_g = \inf_\varphi \frac{\int\limits_M dV (4 \left| \nabla\varphi \right|^2 + R(g)\varphi^2)}{\int\limits_M dV \varphi^2}, \tag{3.24}$$

e.g. see Eq.(7.24) of [50], to be compared against the Yamabe quotient ($p = \frac{2d}{d-2}$ and $\alpha = 4\frac{d-1}{d-2}$) .

$$Y_g = \frac{\int d^d x \sqrt{\hat{g}} \hat{R}(\hat{g})}{\left(\int d^d x \sqrt{\hat{g}} \right)^{\frac{2}{p}}} = \left(\frac{1}{\int_M d^d x \sqrt{g} \varphi^p} \right)^{\frac{2}{p}} \int_M d^d x \sqrt{g} \{\alpha \left(\nabla_g \varphi \right)^2 + R(g)\varphi^2\} \equiv \frac{E[\varphi]}{\|\varphi\|_p^2}$$

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[11]We shall postpone detailed discussion of this topic till Section 6.

also discussed in [50]. Because of similarity of these two quotients the question arises: Can they be equal to each other? The affirmative answer to this question is obtained in Ref.[58]. It can be formulated as

Theorem [58]. Suppose that γ is a conformal class on M which *does not* contain metric of *positive scalar curvature*. Then

$$Y_\gamma = \sup_{g \in \gamma} \lambda_g V(g)^{\frac{2}{d}} \equiv \bar{\lambda}(M), \qquad (3.25a)$$

where $\bar{\lambda}(M)$ is Perelman's $\bar{\lambda}$ invariant. Equivalently,

$$\lambda_g V(g)^{\frac{2}{d}} \leq Y_\gamma, \qquad (3.25b)$$

where $V(g) = \int d^d x \sqrt{\hat{g}}$ is the volume.

The equality happens when g is the Yamabe minimizer. It is metric of unit volume for manifold M of constant scalar curvature (which, according to theorem above, should be negative so that M is hyperbolic 3-manifold). Only for hyperbolic 3-manifolds whose *Yamabe invariant* $\mathcal{Y}^-(M) = \sup_\gamma Y_\gamma$ the gravitational Cauchy problem for source-free gravitational field is well posed [45,46]. For g which is Yamabe minimizer we have $S_{H-E}[\hat{g}] \leq Y_\gamma$. This result can be further extended by noticing that $\mathcal{N} S_{H-E}[\hat{g}] = CS(\mathbf{A})$, where $\mathcal{N}$ is some constant whose value depends upon the explicit form of the gauge field $\mathbf{A}$, and $CS(\mathbf{A})$ is the Chern-Simons invariant to be described in the next section.

To demonstrate that $\mathcal{N} S_{H-E}[\hat{g}] = CS(\mathbf{A})$ it is sufficient to use some results from works by Chern and Simons [59] and by Chern [60]. In [59] it was proven that: a) the Chern-Simons (C-S) functional $CS(\mathbf{A})$ (to be defined in next section) is a conformal invariant of M (Theorem 6.3. of [59]) and, b) that the critical points of $CS(\mathbf{A})$ correspond to 3-manifolds which are (at least locally) conformally flat (Corollary 6.14 of [59]). Subsequently, these results were reobtained by Chern, Ref.[60], in much simpler and more physically suggestive way. In view of these facts, at least for Yamabe minimizers we obtain, $CS(\mathbf{A}) = \mathcal{N} S_Y[\varphi]$, where $\mathcal{N}$ is some constant (different for different gauge groups). That this is the case should come as not too big of a surprise since for Lorentzian 2+1 gravity Witten, Ref.[61], demonstrated the equivalence of the Hilbert-Einstein and C-S functionals without reference to results of Chern and Simons just cited. At the same time, the Euclidean/Hyperbolic 3d gravity was discussed only much more recently, for instance, in the paper by Gukov, Ref.[62]. To avoid duplications we refer our readers to these papers for further details.

4 Floer-style nonperturbative treatment of Y-M fields

4.1 Physical content of the Floer's theory

Striking resemblance between results of nonperturbative treatment of 4-dimensional Y-M fields and two dimensional nonlinear sigma model at the classical level is

well documented in Ref.[63]. Zero curvature equations, e.g. Eq.(2.7), can be obtained either by using the two- dimensional nonlinear sigma model or three-dimensional C-S functional. As discussed in previous section, the self-duality condition for Y-M fields also leads (upon reduction) to zero curvature condition. Since the Ernst equation describing static gravitational (and electrovacuum) fields is obtainable both from conditions of self-duality for the Y-M field and from minimization of 3-dimensional nonlinear sigma model, it follows that 3-d gravitational nonlinear sigma model, Eq.(3.8), contains nonperturbative information about Y-M fields. Furthermore, in view of results of Appendix A, it also should contain information about the static electromagnetic fields, for the combined gravitational and electromagnetic waves and, with minor modifications, for the combined gravitational, electromagnetic and neutrino fields. The nonperturbative treatment of Y-M fields is usually associated either with the instanton or monopole calculations. This observation leads to the conclusion that, at least in some cases, zero curvature equation should carry all nonperurbative information about Y-M fields. This point of view is advocated and developed by Floer [11,21]. Below,we shall discuss Floer's point of view now in the language used in physics literature. For the sake of illustration, it is convenient to present our arguments for Abelian Y-M (that is electromagnetic) fields first.

The action functional S in this case is given by[12]

$$S = \frac{1}{2} \int\limits_0^t dt \int\limits_M dv [\mathbf{E}^2 - \mathbf{B}^2], \qquad (4.1)$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and $\mathbf{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial}{\partial t}\mathbf{A}$, $\varphi \equiv \mathbf{A}_0$. It is known that, at least for electromagnetic waves, it is sufficient to put $\mathbf{A}_0 = 0$ (temporal gauge). In such a case the above action can be rewritten as

$$S[\mathbf{A}] = \frac{1}{2} \int\limits_0^t dt \int\limits_M dv [\dot{\mathbf{A}}^2 - (\boldsymbol{\nabla} \times \mathbf{A})^2], \qquad (4.2)$$

where $\dot{\mathbf{A}} = \frac{\partial}{\partial t}\mathbf{A}$. From the condition $\delta S/\delta \mathbf{A} = 0$ we obtain $\frac{\partial \mathbf{E}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{B}$. The definition of $\mathbf{B}$ guarantees the validity of the condition $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ while from the definition of $\mathbf{E}$ we get another Maxwell equation $\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E}$. The question arises: will these results imply the remaining Maxwell's equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ essential for correct formulation of the Cauchy problem? If such a constraint satisfied at $t = 0$, naturally, it will be satisfied for $t > 0$. Unfortunately, for $t = 0$ the existence of such a constraint does not follow from the above equations and should be introduced as independent. This causes decomposition of the field $\mathbf{A}$ as $\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp$.Taking into account that $\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A}$, we obtain as well $\boldsymbol{\nabla}\cdot(\mathbf{E}_\parallel +\mathbf{E}_\perp)$. Then, by design $\boldsymbol{\nabla}\cdot\mathbf{E}_\perp = 0$, while $\boldsymbol{\nabla}\cdot\mathbf{E}_\parallel$ remains to be defined by the initial and boundary data. Because of this, it is always possible to choose $\mathbf{A}_\parallel = 0$ and to use only $\mathbf{A}_\perp$ for description of the field propagation [64]. Hence,

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[12]Up to an unimportant scale factor.

the action functional S can be finally rewritten as

$$S[\mathbf{A}_\perp] = \frac{1}{2} \int\limits_0^t dt \int\limits_M dv [\dot{\mathbf{A}}_\perp^2 - (\boldsymbol{\nabla} \times \mathbf{A}_\perp)^2]. \tag{4.3}$$

In such a form it can be used as action in the path integrals, e.g. see Ref.[64], page 152, describing free electromagnetic field. Such path integral can be evaluated both in Minkowski and Eucldean spaces by the saddle point method. There is, however, a closely related method more suitable for our purposes. It is described in the monograph by Donaldson, Ref.[11]. Following this reference, we replace time variable t by $-i\tau$ in the functional $S[\mathbf{A}_\perp]$ .Consider now this replacement in some detail. We have[13]

$$\frac{1}{2} \int\limits_0^T d\tau \int\limits_M dv [\dot{\mathbf{A}}_\perp^2 + (\boldsymbol{\nabla} \times \mathbf{A}_\perp)^2]$$

$$= \frac{1}{2} \int\limits_0^T d\tau (\int\limits_M dv [[\dot{\mathbf{A}}_\perp + (\boldsymbol{\nabla} \times \mathbf{A}_\perp)]^2 - \frac{\partial}{\partial \tau}(\mathbf{A}_\perp \cdot \boldsymbol{\nabla} \times \mathbf{A}_\perp)]). \tag{4.4}$$

Since variation of $\mathbf{A}_\perp$ is fixed at the ends of τ integral, the last term can be dropped so that we are left with the condition

$$\frac{\partial}{\partial \tau} \mathbf{A}_\perp = -\mathbf{B}_\perp \tag{4.5}$$

extremizing the Euclidean action $S_E[\mathbf{A}_\perp]$. The above results are transferable to the non Abelian Y-M field by continuity and complementarity. Since in the Abelian case fields $\mathbf{E}$ and $\mathbf{B}$ are dual to each other, by applying the curl operator to both sides of Eq.(4.5) (and removing the subscript $\perp$) we obtain the equivalent form of self-duality equations in accord with those on page 33 of Ref.[6]. This calculation provides an independent check of Donaldson's method of computation. Since the (anti)self-duality condition in the Abelian case can be written as $\mathbf{B} = \mp\mathbf{E}$ [9].and since $\mathbf{E} = -\frac{\partial}{\partial \tau}\mathbf{A}$, we conclude that Eq.(4.5) is the self-duality equation. This conclusion is immediately transferable to the non Abelian Y-M case where the analog of Eq. (4.5) is

$$\frac{\partial}{\partial \tau}\mathbf{A} = *\mathbf{F}(\mathbf{A}(\tau)), \tag{4.6}$$

in accord with Floer. The symbol * denotes the Hodge star operation in 3 dimensions. Following Donaldson [11] this result should be understood as follows. Introduce a connection matrix $\mathbf{A} = A_0 d\tau + \sum\limits_{i=1}^3 A_i dx_i$ such that both A_0 and A_i depend upon all four variables τ, x_1, x_2 and x_3. In the temporal gauge A_0 should

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[13]We shall assume (without loss of generality) that $\dot{\mathbf{A}}_\perp$ is collinear with $\mathbf{A}_\perp$.

be discarded so that τ becomes a parameter in the remaining $A_i's$. Evidently, it can be associated with the spectral parameter (e.g. see previous section). The Hodge star operator in Eq.(4.6) is needed to make this equation as an equation for one-forms The obtained results fit nicely into Cauchy formulation of dynamics of both Y-M and gravity. Indeed, under conditions analogous to that discussed in [45,46] the space-time (4-manifold) is decomposable into direct product $M \times R$ (a trivial fiber bundle) in such a way that all differential operations acting on 4-manifold are been projected down to 3-manifold M. This is essential part of Floer's theory. Furthermore, since $\delta CS(\mathbf{A})/\delta\mathbf{A} = \mathbf{F}(\mathbf{A})$ the above Eq.(4.6) can be equivalently rewritten as

$$\frac{\partial}{\partial\tau}\mathbf{A} = *[\delta CS(\mathbf{A})/\delta\mathbf{A}] \tag{4.7}$$

so that the Chern-Simons functional is playing a role of a "Hamiltonian" in Eq.s(4.7). From the theory of dynamical systems it follows then that the dynamics is taking place between the points of equilibria defined by zero curvature condition $\mathbf{F}(\mathbf{A}) = 0$. At the same time, using our work, Ref.[50], it is easily recognize Eq.(4.7) as an equation for the gradient flow, e.g. see Eq.s(3.14). For the sake of space we shall not discuss this topic any further. Interested readers are encouraged to consult Ref.[65]. For supersymmetric Y-M fields participating in Seiberg-Witten theory the gradient flow equations are discussed in detail in Ref.[66]

The mechanical system described by Eq.(4.7) should be eventually quantized. Since the quantization procedure is outlined in Ref.[67], to avoid duplications, we shall concentrate attention of our readers on aspects of Floer's theory not covered in [67] but still relevant to this paper. To do so, we follow Donaldson [11]. This is accomplished in several steps.

First, in the previous section we noticed that the axially symmetric self-dual solution for Y-M fields does not depend on x_0 (or τ) variable. Therefore, if such solution is substituted back into Y-M functional, it produces divergent result. Although the cure for this issue is discussed in subsection 4.4, in this subsection we provide needed background. For this purpose, following Ref.[68] we consider the Y-M action $S[\mathbf{F}]$ for the pure Y-M field[14]

$$S[\mathbf{F}] = -\frac{1}{8} \int_{\mathbf{R}^4} d^4 x tr(F_{\mu\nu}F_{\mu\nu}). \tag{4.8}$$

The duality condition[15] $*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ allows us then to rewrite this action as follows

$$S[\mathbf{F}] = -\frac{1}{16} \int_{\mathbf{R}^4} d^4 x [tr((F_{\mu\nu} \mp *F_{\mu\nu})(F_{\mu\nu} \mp *F_{\mu\nu})) \pm 2tr(F_{\mu\nu} * F_{\mu\nu})] \tag{4.9}$$

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[14] Strictly following notations of Ref.[68] we do not indicate that in general the integration should be made over some 4-manifold $\mathcal{M}$. In physics literature, and in Eq.s(2.11), it is assumed that we are dealing with $\mathbf{R}^4$ (or S^4 upon compactification). In Floer's theory it is essential that the 4-manifold is decomposable as $M \times R$. This decomposition should be treated with care as described in the Donaldson's book [11]

[15] With the convention that $\varepsilon_{1234} = -1$.

since $tr(F_{\mu\nu}F_{\mu\nu}) = tr(*F_{\mu\nu} * F_{\mu\nu})$. The winding number N for SU(2) gauge field is defined as[16]

$$N = -\frac{1}{8\pi^2} \int_{\mathbf{R}^4} d^4x\, tr(F_{\mu\nu} * F_{\mu\nu}) \equiv -\frac{1}{8\pi^2} \int_{\mathbf{R}^4} tr(F_{\mu\nu} \wedge F_{\mu\nu}) \qquad (4.10)$$

so that use of this definition in Eq.s(4.8),(4.9) produces

$$S[\mathbf{F}] \geq \pi^2 |N| \qquad (4.11)$$

with the equality taking place when the (anti) self-duality condition (e.g. see Eq.(2.10)) holds. In such a case the saddle point action is becoming just a winding number (up to a constant).

Second, if our space-time 4-manifold $\mathcal{M}$ can be decomposed as $M \times [0,1]$, the following identity can be used [11]

$$\int_{M\times[0,1]} tr(F_{\mu\nu} \wedge F_{\mu\nu}) = \int_M tr(\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}) \doteq CS(\mathbf{A}). \qquad (4.12)$$

Here the symbol $\doteq$ means "up to a constant". The decomposition $M \times [0,1]$ reflects the fact that the C-S functional is defined up to mod $\mathbf{Z}$. This ambiguity can be removed if we agree to consider C-S functional as a quotient $\mathbf{R}/\mathbf{Z}$. Accordingly, this allows us to replace $M \times R$ by $M \times [0,1]$. Details can be found in Ref.[11]. Thus, one way or another the winding number N in Eq.(4.10) can be replaced by the Chern-Simons functional.

Third, since the equation of motion for the C-S functional is zero curvature condition $\mathbf{F} = 0$, i.e.

$$d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = 0, \qquad (4.13)$$

implying that the connection $\mathbf{A}$ is flat, we can use this result in Eq.(4.12) in order to rewrite it as (e.g. for SU(2))

$$\frac{1}{8\pi^2} \int_M tr(\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}) = -\frac{1}{24\pi^2} \int_M tr(\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}). \qquad (4.14)$$

For other groups the prefactor and the domain of integration will be different in general.

Fourth, zero curvature Eq.(4.13) involves connections which are functions of three arguments and a spectral/time parameter. In such setting minimization of Y-M functional is not divergent in view of Eq.(4.11).

Fifth, the obtained result, Eq.(4.14), coincides with that known for the winding number for SU(2) instantons in physics literature[8,19] where it was obtained with help of entirely different arguments. It should be noted though that in spite of apparent simplicity of these results, actual calculations of C-S functionals (invariants) for different 3-manifolds are, in fact, very sophisticated [69,70]. In accord with Floer and Ref.[67], we conclude that nonperturbatively

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[16]We follows notations of Ref.[68] in which $\mathbf{R}^4$ is actually standing for $S^4 \in SU(2)$

the 4-dimensional pure Y-M quantum field theory is a topological field theory of C-S type.

Sixth, the isomorphism noticed by Louis Witten acquires now natural explanation. It becomes possible in view of results just presented, on one hand, and the fact that $\mathcal{N}S_{H-E}[\hat{g}] = CS(\mathbf{A})$ (previous section), on another. For fields with axial symmetry, equations of motion, Eq.(4.13), for gravity and Y-M fields coincide.

Seventh, the instantons in Floer's theory are *not* the same as considered in physics literature [8,19]. To understand this, we must take into account that in Floer's theory manifolds under consideration are 4-manifolds $\mathcal{M}$ with tubular ends. Such manifolds are complete Riemannian manifolds with finite number of tubular ends made of half tubes $(0, \infty)$ so that locally each such manifold looks like $U_i = L_i \times (0, \infty)$ with L_i being a compact 3-manifold (called a "crossection of a tube") and i numbering the tubes. The closure of $\mathcal{M} \backslash \bigcup_{i=1}^{n} U_i$ is a compact manifold with boundary. If the crossection is S^3, then U is conformally equivalent to a punctured ball $B^4 \backslash \{0\}$. This implies that a manifold $\mathcal{M}$ with tubular ends is conformally equivalent to a punctured manifold $\tilde{\mathcal{M}} \backslash \{p_1, ..., p_n\}$ where $\tilde{\mathcal{M}}$ is compact. The instanton moduli problem for $\mathcal{M}$ is equivalent to that for the punctured manifold [11]. Recall that the moduli space of instantons is defined as set of solutions of anti self-dual equations modulo gauge equivalence.

Being armed with these definitions and taking into account that the (anti) self-duality Eq.(4.7) we can interpret the instanton as a path connecting one flat connection $\mathbf{F} = 0$ at "time" $\tau = -\infty$ with another flat connection at "time" $\tau = \infty$ [11]. It is permissible for the path to begin at one flat connection, to wind around a tube (modulo gauge equivalence) and to end up at the same flat connection, Ref.[11], page 22. Evidently, this case involves 4-manifolds with just one tubular end. Physically, each flat connection $\mathbf{F} = 0$ represents the vacuum state so that the instantons discussed in the Introduction should be connecting different vacua. In this sense there is a difference between the interpretation of instantons in mathematics and physics literature. As in the case of standard quantum mechanics, only imposition of some additional physical constraints permits us to select between all possible solutions only those which are physically relevant. In the present context it is known that all exactly integrable systems are described by the zero curvature equation $\mathbf{F} = 0$ [5,6]. It is also known that differences between these equations are caused in part by differences in a way the spectral parameter enters into these equations. Since for the Floer's instantons $\mathbf{F} \neq 0$, it means that the curvature $\mathbf{F}$ should be parametrized in such a way that the "time" parameter should become a spectral parameter when $\mathbf{F} = 0$. In this work we do not investigate this problem[17]. Instead, we shall focus our attention on different vacua, that is on different (knot-like) solutions of zero curvature equation $\mathbf{F} = 0$.[18]

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[17]See Ref.[7] for introduction into this topic.

[18]A complement of each knot in S^3 is 3-manifold. Floer's instantons are in fact connecting various three-manifolds. These 3- manifolds (with tubular ends) should be glued together to form $\mathcal{M}$.The gluing procedure is extremely delicate mathematical operation [11]. It is above

4.2 The Faddeev-Skyrme model and vacuum states of the Y-M functional

In the light of results just presented, we would like to argue that the F-S model is indeed capable of representing the vacuum states of pure Y-M fields. For this purpose it is sufficient to recall the key results of the paper by Auckly and Kapitansky [71]. These authors were able to rewrite the Faddeev functional

$$E[\mathbf{n}] = \int_{S^3} dv\{|d\mathbf{n}|^2 + |d\mathbf{n} \wedge d\mathbf{n}|^2\} \qquad (4.15)$$

in the equivalent form given by

$$E_\varphi[a] = \int_{S^3} dv\{|D_{\mathbf{a}}\boldsymbol{\varphi}|^2 + |D_{\mathbf{a}}\boldsymbol{\varphi} \wedge D_{\mathbf{a}}\boldsymbol{\varphi}|^2\}. \qquad (4.16)$$

In this expression, the covariant derivative $D_a\boldsymbol{\varphi} = d\boldsymbol{\varphi} + [\mathbf{a}, \boldsymbol{\varphi}]$. Evidently, $E_\varphi[\mathbf{a}]$ acquires its minimum when $\boldsymbol{\varphi} = \mathbf{a}$ and the connection becomes flat (that is covariant derivative becomes zero). Since this result is compatible with those discussed in previous subsection, it implies that, indeed, Faddeev's model can be used for description of vacuum states for pure Y-M fields. The only question remains: Is this model the only model describing QCD vacuum? In view of Eq.s (3.18),(3.19) it should be clear that this is not the case. The full explanation is given below, in Sections 5,6. In addition, the disadvantage of the F-S model as such (that is without modifications) lies in the absence of gap upon its quantization as was recognized already by Faddeev and Niemi in Ref.[25]. In Sections 5,6 we shall eliminate this deficiency in a way different from that described in the Introduction (e.g. in Ref.s[25,26]). In the meantime, we would like to find the place for monopoles in our calculations.

4.3 Monopoles and the Ernst equation

4.3.1 Monopoles versus instantons

To introduce notations and for the sake of uninterrupted reading, we need to describe briefly the alternative point of view at the results of previous subsection. For this purpose, following Manton [49], we need to make a comparison between the Lagrangians for SU(2) Y-M and the Y-M-Higgs fields described respectively by

$$\mathcal{L}_{Y-M} = -\frac{1}{4}tr(F_{\mu\nu}F^{\mu\nu}) \qquad (4.17)$$

and

$$\mathcal{L}_{Y-M-H} = -\frac{1}{4}tr(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) - \frac{1}{2}tr(D_\mu\boldsymbol{\Phi}\cdot D^\mu\boldsymbol{\Phi}) - \frac{\lambda}{2}(1 - \boldsymbol{\Phi}\cdot\boldsymbol{\Phi})^2 \qquad (4.18)$$

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the level of rigor of this paper. To imagine the connected sum of knots [20] is much easier than the connected sum of 3-manifolds. This sum has physical meaning discussed in Section 6.

with covariant derivative for the Higgs field defined as $D_\mu \mathbf{\Phi} = \partial_\mu \mathbf{\Phi} + [\mathbf{A}_\mu, \mathbf{\Phi}]$ and with connection $\mathbf{A}_\mu$ used to define the Y-M curvature tensor $\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu]$, provided that $\mathbf{\Phi} = \Phi^a t^a$, $\mathbf{A}_\mu = A_\mu^a t^a$, and $[t^a, t^b] = -2\varepsilon_{abc} t^c$. Now the self-duality condition $\mathbf{F} = *\mathbf{F}$ can be equivalently rewritten as $\mathbf{F}_{ij} = -\varepsilon_{ijk} \mathbf{F}_{k0}$ with indices i, j, k running over 1,2,3. Incidentally, in the temporal gauge this result is equivalent to Floer's Eq.(4.6) Consider now the limit $\lambda \to 0$ in Eq.(4.18). In the Minkowski spacetime the field equations originating from the Y-M-Higgs Lagrangian can be solved by using the Bogomolny ansatz equations $\mathbf{F}_{ij} = -\varepsilon_{ijk} D_k \mathbf{\Phi}$ in which $\mathbf{A}_0 = 0$ (temporal gauge). Instead of imposing the temporal gauge condition, we can identify the Higgs field $\mathbf{\Phi}$ with $\mathbf{A}_0$ so that the Bogomolny equations read now as follows:

$$\mathbf{F}_{ij} = -\varepsilon_{ijk} D_k \mathbf{A}_0. \tag{4.19}$$

Bogomolny demonstrated that the Prasad-Sommerfield monopole solution can be obtained using Eq.(4.19). Thus, any static (that is time-independent) solution of self-duality equations is leading to Bogomolny-Prasad-Sommerfield (BPS) monopole solution of the Y-M fields, provided that we interpret the component $\mathbf{A}_0$ as the Higgs field. Suppose now that there is an axial symmetry. Forgacs, Horvath and Palla [72] (FHP) demonstrated equivalence of the set of axially symmetric Bogomolny Eq.s (4.19) to the Ernst equation. The static monopole solution is time-independent self-dual gauge field. Because of this time independence, its four-dimensional action is *infinite* (because of the time translational invariance) while that for instantons *is finite*. Furthermore, the boundary conditions for monopoles and instantons are different. The infinity problem for monopoles can be cured somehow by considering the monopole dynamics [68] but this topic at this moment "is more art than science", e.g. read [68], page 309. For the same reason we avoid in this section talking about dyons (pseudo particles having both electric and magnetic charge). Hence, we would like to conclude our discussion with description of more mathematically rigorous treatments. By doing so we shall establish connections with results presented in previous sections

4.4 Calorons.

Calorons are instantons on $R^3 \times S^1$. From this definition it follows that, physically, these are just instantons at finite temperature[19]. Calorons are related to both instantons on $\mathbf{R}^4$ (or S^4) and monopoles on $\mathbf{R}^3$ (or S^3). Heuristically, the large period calorons are instantons while the small period calorons are monopoles [73,74]. These results do not account yet for the fact that both the Y-M action and the self-duality equations are conformally invariant. Atiyah [75]. noticed that the Euclidean metric can be represented either as

$$ds_E^2 = \left(dx^1\right)^2 + \left(dx^3\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2 \tag{4.20a}$$

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[19]This explains the word "caloron".

or as

$$ds^2 = \frac{r^2}{R^2}[R^2(\frac{\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dr\right)^2}{r^2}) + R^2 d\varphi^2] \qquad (4.20b)$$

with R being some constant. The above representation involves polar r, φ coordinates in the (x^3, x^4) plane thus implying some kind of axial symmetry. Since self-duality equations are conformally invariant, the prefactor $\frac{r^2}{R^2}$ can be dropped so that the Euclidean space $\mathbf{R}^4$ becomes conformally equivalent to the product $\mathbf{H}^3 \times S^1$. For such manifold the constant scalar curvature of the hyperbolic 3-space $\mathbf{H}^3$ is $-1/R^2$. Furthermore, the remaining term represents the metric on a circle of radius R. Following Ref.[73], let (x^1, x^2, x^3) be coordinates for the hyperbolic ball model of $\mathbf{H}^3$ so that the radial coordinate be $\mathcal{R} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Let $0 \leq \mathcal{R} \leq R$. Let τ be a coordinate on S^1 with period β, then the metric on $\mathbf{H}^3 \times S^1$ can be represented as

$$ds_H^2 = d\tau^2 + \Lambda^2(d\mathcal{R}^2 + \mathcal{R}^2 d\Omega^2), \qquad (4.21a)$$

where $\Lambda = (1 - \mathcal{R}/R)^{-1}$ and $d\Omega^2$ is the metric on 2-dimensional sphere. If we introduce an auxiliary coordinate $\mu = (R/2)\operatorname{arctanh}(\mathcal{R}/R)$, and complex coordinate $z = \mu + i\tau$, the above hyperbolic metric can be rewritten as

$$ds_H^2 = d\tau^2 + d\mu^2 + \Xi^2 d\Omega^2 \qquad (4.21b)$$

with $\Xi = (R/2)\sinh(2\mu/R)$. By analogy with transition from Eq.(4.20a) to (4.20b) we can proceed as follows. Let $r = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$ with (y^1, y^2, y^3, y^0) being coordinates on $\mathbf{R}^4$. By letting $t = y^0$ the Euclidean metric can be written as usual, i.e.

$$ds_E^2 = dt^2 + dr^2 + r^2 d\Omega^2 \qquad (4.22a)$$

so that

$$ds_H^2 = \xi^2 ds_E^2, \qquad (4.22b)$$

with $\xi = (R/2)[\cosh(2\mu/R) + \cos(2\tau/R)]$. This correspondence between $\mathbf{R}^4 \smallsetminus \mathbf{R}^2$ and $\mathbf{H}^3 \times S^1$ is made with help of the mapping $w = \tanh(z/R)$ (with $w = r + it$ and $\beta = \pi R$). Let $\mathcal{M} = \mathbf{H}^3 \times S^1$ (or $\mathbf{H}^3 \times R$) then, in view of conformal invariance, we can rewrite Eq.(4.8) as

$$S[\mathbf{F}] = -\frac{1}{8} \int_{\mathcal{M}} tr(F_{\mu\nu} F^{\mu\nu}) \Xi^2 d\tau d\mu d\Omega. \qquad (4.23)$$

We have to rewrite the winding number, Eq.(4.10), accordingly. Since it is a topological invariant, this means that the self-duality equations must be adjusted accordingly. For instance, for the *hyperbolic calorons* on $\mathbf{H}^3 \times S^1$ the self-duality equation reads

$$F_{0i} = \frac{1}{2\Lambda}\varepsilon_{ijk}F_{jk}. \qquad (4.24)$$

The action $S[\mathbf{F}]$ now is finite with $tr(F_{\mu\nu}F^{\mu\nu}) \to 0$ when $\mu \to \infty$. For hyperbolic instantons we have finite action with $tr(F_{\mu\nu}F^{\mu\nu}) \to 0$ when $\mu^2 + \tau^2 \to \infty$.

The results just described match nicely with the results by Witten [76] on Euclidean SU(2) instantons invariant under the action of SO(3) Lie group. His results will be discussed in detail in the next section. Notice, that Euclidean metric, Eq.(4.22a), becomes that for $\mathbf{H}^2 \times S^2$ if we rewrite it as

$$ds_E^2 = r^2 \big(\frac{dt^2 + dr^2}{r^2} + d\Omega^2 \big) \qquad (4.25a)$$

and, as before, we drop the conformal factor r^2 so that it becomes

$$ds_H^2 = \frac{dt^2 + dr^2}{r^2} + d\Omega^2. \qquad (4.25b)$$

Interestingly enough, that results by Witten initially developed for $\mathbf{H}^2 \times S^2$ can be also used without change for $\mathbf{H}^3 \times S^1$ and $\mathbf{H}^3 \times \mathbf{R}$ since the action of SO(3) pulls back to these manifolds [73]. *This fact is of importance since such an extension makes his results compatible with both Floer's method of calculations for Y-M fields and with results of Section 3.* Omitting all details, the action, Eq.(4.23), is reduced to that known for two dimensional Abelian Ginzburg-Landau (G-L) model "living" on the hyperbolic 2 manifold X coordinatized by μ and τ with the metric

$$ds_H^2 = \frac{d\mu^2 + d\tau^2}{\Xi^2}. \qquad (4.26)$$

Explicitly, such G-L action functional S_{G-L} is given by [73]

$$S_{G-L} = \frac{\pi}{2} \int_X d\tau d\mu [\Xi^2 (\boldsymbol{\nabla} \times \mathbf{A})^2 + 2\, |(\boldsymbol{\nabla} + i\mathbf{A})\phi|^2 + \Xi^{-2}(1 - |\phi|^2)] \qquad (4.27)$$

with $\mathbf{A}$ and ϕ being respectively the Abelian gauge and the Higgs fields, $\phi = \phi_1 + i\phi_2$ so that $|\phi|^2 = \phi_1^2 + \phi_2^2$. This functional is obtained upon substitution of solution of the self-duality equations into the Y-M action functional, Eq.(4.23). We refer our readers to the original paper, Ref.[73] for details. In the limit $\beta \to \infty$ the above functional coincides with that obtained by Witten [76]. The self-duality equations obtained by Witten describe instantons which lie along the fixed axis while Fairlie, Corrigan, 't Hooft, and Wilczek [77] developed an ansatz (CFtHW ansatz) for the self-duality equations producing instantons at arbitrary locations. Manton [78] demonstrated that Witten's and CFtHW multi-instanton solutions are gauge equivalent while Harland [73,74] demonstrated how these instantons and monopoles can be obtained from calorons in various limits. The obtained results provide needed background information for solution of the gap problem. This solution is discussed in the next section.

5   Solution of the gap problem

5.1   Idea of the proof

By cleverly using symmetry of the problem Witten [76] reduced the non Abelian Y-M action functional to that for the Abelian G-L model "living" in the hyperbolic plane. This is one of examples of the Abelian reduction of QCD discussed

in our paper, Ref.[79]. Vortices existing in the G-L model could be visualized as made of some two-dimensional surfaces (closed strings) living in the ambient space-time. These are known as Nambu-Gotto strings. Their treatment by Polyakov [80] made them to exist in spaces of higher dimensionality. In order for them to be useful for QCD, Polyakov suggested to modify string action by adding an extra (rigidity) term into string action functional. By doing so the problem was created of reproducing Polyakov rigid string model from QCD action functional. The latest proposal by Polyakov [81] involves consideration of spin chain models while that by Kondo [82] involves the F-S model derived directly from QCD action functional. As explained in[79], in the case of scattering processes of high energy physics one is confronted essentially with the same combinatorial problems as were encountered at the birth of quantum mechanics. In Ref.[83] we explained in detail why Heisenberg's (combinatorial) method of developing quantum mechanical formalism is superior to that by Schrödinger. In Ref.[74] using these general results we demonstrated how the combinatorial analysis of scattering data leads to spin chain models as microscopic models describing excitation spectrum of QCD. Thus, the mass gap problem can be considered as already solved in principle. Nevertheless, in [**94**] such a solution is obtained "externally", just based on the rules of combinatorics. As with quantum mechanics, where atomic model is used to test Heisenberg's ideas, there is a need to reproduce this combinatorial result "internally" by using microscopic model of QCD. For this purpose, we shall use the G-L functional, Eq.(4.27). By analogy with the flat case, we expect that it can be rewritten in terms of interacting vortices. In the present case, in view of Eq.(4.26), vortices "live" not in the Euclidean plane but in 3+1 Minkowski space-time. This is easy to understand if we recall the SO(3)⇌SU(2) correspondence and take into account the analogous correspondence between SU(1,1) and SO(2,1).

Within such a picture it is sufficient to look at evolution dynamics of the individual vortex. Typically, it is well described by the dynamics of the continuous Heisenberg spin chain model [84,85] in Euclidean space. In the present case, this formalism should be extended to the Minkowski space and, eventually, to hyperbolic space (that is to the case of Abelian model discovered by Witten). Details of such an extension are summarized in Appendix B. After that, the next task lies in connecting these results with the Ernst equation. In the next subsection we initiate this study.

5.2 Heisenberg spin chain model and the Ernst equation

For the sake of space, this subsection is written under assumption that our readers are familiar with the book "Hamiltonian methods in the theory of solitons" [86] (or its equivalent) where all needed details can be found. The continuous XXX Heisenberg spin chain is described with help of the spin vector[20]

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[20]In compliance with [86] we suppress the time-dependence.

$\vec{S}(x) = (S_1(x), S_2(x), S_3(x))$ restricted to live on the unit sphere S^2 :

$$\vec{S}^2(x) = \sum_{i=1}^{3} S_i^2(x) = 1 \tag{5.1}$$

while obeying the equation of motion

$$\frac{\partial}{\partial t}\vec{S} = \vec{S} \times \frac{\partial^2}{\partial x^2}\vec{S} \tag{5.2}$$

known as the Landau-Lifshitz (L-L) equation. By introducing matrices $U(\lambda)$ and $V(\lambda)$ via

$$U(\lambda) = \frac{\lambda}{2i}S, \ V(\lambda) = \frac{i\lambda^2}{2}S + \frac{\lambda}{2}S\frac{\partial}{\partial x}S, S = \vec{S} \cdot \vec{\sigma} \tag{5.3}$$

so that σ_i is one of Pauli's spin matrices and λ is the spectral parameter and requiring that $S^2 = I$, where I is the unit matrix, the zero curvature condition

$$\frac{\partial}{\partial t}U - \frac{\partial}{\partial x}V + [U, V] = 0 \tag{5.4}$$

is obtained. With account of the constraint $S^2 = I$ it can be converted into equation

$$\frac{\partial}{\partial t}S = \frac{1}{2i}[S, \frac{\partial^2}{\partial x^2}S] \tag{5.5}$$

equivalent to Eq.(5.2). The correspondence between Eq.s(5.2) and (5.5) can be made for $S(x,t)$ matrices of arbitrary dimension.

Having in mind Witten's result [76], we want now to extend these Euclidean results to the case of noncompact Heisenberg spin chain model "living" either in Minkowski or hyperbolic space. In doing so we follow, in part, Ref.[56] and Appendix B. For this purpose we need to remind our readers some facts about the Lie group SU(1,1). Since this group is related to SO(2,1), very much like SU(2) is related to SO(3), we can proceed by employing the noticed analogy. In particular, since $S = \vec{S} \cdot \vec{\sigma}$, we can preserve this relation by writing now $S = \vec{S} \cdot \vec{\tau}$. Using this result we obtain,

$$S = \begin{pmatrix} S^z & iS^- \\ iS^+ & -iS^z \end{pmatrix} \in su(1,1), \ S^{\pm} = S^x \pm iS^y, \tag{5.6}$$

where the form of matrices generating su(1,1) Lie algebra is similar to that for Pauli matrices. This time, however, $\det S = -1$ even though $S^2 = I$. Explicitly, $(S^z)^2 - (S^x)^2 - (S^y)^2 = 1$, that is the motion is taking place on the unit pseudosphere $S^{1,1}$. Matrices τ_i generating $su(1,1)$ are fully characterized by the following two properties

$$tr(\tau_\alpha \tau_\beta) = 2g_{\alpha\beta} \ , \ [\tau_\alpha, \tau_\beta] = 2if_{\alpha\beta\gamma}\tau_\gamma; \ g_{\alpha\beta} = diag(-1, -1, 1); \alpha, \beta, \gamma = 1, 2, 3 \tag{5.7}$$

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with $f_{\alpha\beta\gamma}$ being structure constants for $su(1,1)$ algebra. An analog of the equation of motion, Eq.(5.5), now reads

$$\frac{\partial}{\partial t}S^\alpha = \sum_{\beta,\gamma} f^{\alpha\beta\gamma} S_\beta \frac{\partial^2}{\partial x^2} S_\gamma.$$

(5.8)

If we defines the Poisson brackets as $\{S^\alpha(x), S^\beta(y)\} = -f^{\alpha\beta\gamma} S^\gamma(x)\delta(x-y)$, then the above equation of motion can be rewritten in the Hamiltonian form

$$\frac{\partial}{\partial t}S^\alpha = \{H, S^\alpha\},$$

(5.9)

provided that the Hamiltonian H is given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx\, (\nabla_x S^\alpha)\, g_{\alpha\beta}\, (\nabla_x S^\beta) \equiv \frac{1}{4}tr \int_{-\infty}^{\infty} dx\, (\nabla_x S)^2.$$

(5.10)

Since now the motion takes place on pseudosphere $\check{S}^2$, it is convenient to introduce the pseudospherical coordinates by analogy with spherical, e.g.

$$S^x(x,t) = \sinh\theta(x,t)\cos\varphi(x,t), S^y(x,t) = \sinh\theta(x,t)\sin\varphi(x,t), S^z(x,t) = \cosh\theta(x,t).$$

(5.11)

Also, by analogy with spherical case we can use the stereographic projection : from pseudosphere to hyperbolic plane. Recall [**102**], that in the case of a sphere S^2 the inverse stereographic projection: from complex plane $\mathbf{C}$ to S^2 is given by

$$S^+ = \frac{2z}{1+|z|^2}, S^- = \frac{2z^*}{1+|z|^2}, S^z = \frac{1-|z|^2}{1+|z|^2}.$$

(5.12)

The mapping from $\mathbf{C}$ to $\mathbf{H}^2$ is obtained with help of Eq.(5.12) in a straightforward way as

$$S^+ = \frac{2\xi}{1-|\xi|^2}, S^- = \frac{2\xi^*}{1-|\xi|^2}, S^z = \frac{1+|\xi|^2}{1-|\xi|^2}.$$

(5.13)

Using this correspondence the equations of motion, Eq.(5.10), rewritten in terms of ξ and ξ^* variables (while keeping in mind that they are parametrized by x and t) are given by

$$i\frac{\partial}{\partial t}\xi + \frac{\partial^2}{\partial x^2}\xi + 2\frac{\xi^*}{1-|\xi|^2}\left(\frac{\partial}{\partial x}\xi\right)^2 = 0.$$

(5.14)

In the static (t−independent) case the above equation is reduced to

$$(|\xi|^2 - 1)\nabla_x^2\xi = 2\xi^*\,(\nabla_x\xi)^2$$

(5.15)

easily recognizable as the Ernst equation. In his paper, Ref. [42], Ernst used variational principle applied to the functional Eq.(3.18). From Appendix B we

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know that both the L-L equation and its hyperbolic version describe the motion of (could be knotted) vortex filament. Because of this, the functional, Eq.(3.18), should undergo the same reduction as was made in going from Eq.(B1.a) to (B1.b). Explicitly, this means that the functional, Eq.(3.18), should be reduced in such a way that the Hamiltonian, Eq.(5.10), should be replaced by

$$H = -2 \int_{-\infty}^{\infty} dx \frac{|\nabla_x \xi|^2}{(1 - |\xi|^2)^2},$$

(5.16)

where the sign in front is chosen in accord with Ref.[87] and our Eq.(3.22). The Hamiltonian equation of motion, Eq.(5.9), reproducing Eq.(5.14) can be obtained if the Poisson bracket is defined as by $\{\xi(x), \xi^*(y)\} = (1 - |\xi|^2)^2 \delta(x - y)$. The obtained results set up the stage for quantization. It will be discussed in subsection 5.4. In the meantime, we need to connect results of Witten's work, Ref.[76], with those we just obtained.

5.3   From Abelian Higgs to Heisenberg spin chain model

5.3.1   The Abelian Higgs model

The work by Witten [76] had been further analyzed in the paper by Forgacs and Manton [88]. The major outcome of their work lies in demonstration of uniqueness of the self-duality ansatz proposed by Witten. The self-duality equations obtained in Witten's work are reduced to the system of three coupled equations describing interaction between the Abelian Y-M and Higgs fields

$$\partial_0 \varphi_1 + A_0 \varphi_2 = \partial_1 \varphi_2 - A_1 \varphi_1,$$

(5.17a)

$$\partial_1 \varphi_1 + A_1 \varphi_2 = -(\partial_0 \varphi_2 - A_0 \varphi_1),$$

(5.17b)

$$r^2(\partial_0 A_1 - \partial_1 A_0) = 1 - \varphi_1^2 - \varphi_2^2.$$

(5.17c)

To analyze these equations, we recall that the original self-duality equations for the Y-M fields are conformally invariant. We can take advantage of this fact now by temporarily dropping the conformal factor r^2 in Eq.(5.17c). Then, the above equations become the Bogomolny equations for the flat space Abelian Higgs model, e.g. for the model described by the action functional, Eq.(4.24), with the conformal factor $\Xi = 1$ [89]. Such obtained equations contain all information about the Abelian Higgs model and, hence, they are equivalent to this model. It is of importance for us to demonstrate this explicitly for both Euclidean and hyperbolic spaces. For this purpose we introduce a covariant derivative $D_\mu = \partial_\mu - iA_\mu$, $\mu = 0, 1$, and the complex field $\phi = \phi_1 + i\phi_2$. Consider the Bogomolny equation following [68]:

$$D_0 \phi + iD_1 \phi = 0.$$

(5.18)

Using the above definitions straightforward computation reproduces Eq.s(5.17a,b). These equations can be used to obtain

$$r^2 (D_0 - iD_1)(D_0 + iD_1)\phi = 0$$

(5.19)

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implying

$$r^2(D_0 D_0 + D_1 D_1)\phi = -ir^2[D_0, D_1]\phi = -r^2 \left(\partial_0 A_1 - \partial_1 A_0\right)\phi = -(1 - \varphi_1^2 - \varphi_2^2)\phi, \quad (5.20)$$

where the last equality was obtained with help of Eq.(5.17c). Evidently, the equation

$$(D_0 D_0 + D_1 D_1)\phi + \frac{1}{r^2}(1 - \varphi_1^2 - \varphi_2^2)\phi = 0 \qquad (5.21)$$

is one of the equations of "motion" for the G-L model on $\mathbf{H}^2$, e.g. see Ref.[89](Eq.(11.3) page 98). The second is the Ampere's equation

$$\varepsilon_{\mu\nu}\partial_\mu(r^2 B) = i(\phi \bar{D}_\nu \bar{\phi} - \bar{\phi} D_\nu \phi) \qquad (5.22)$$

with the "magnetic field" $B = \partial_0 A_1 - \partial_1 A_0$. Details of derivation are given in Ref.[68], pages 198-199. Eq.(5.22) also coincides with that given in the book by Taubs and Jaffe, Ref.[**105**] (Eq.(11.3) page 98).

Corollary 1. *Since both equations can be obtained by mimization of the functional, Eq.(4.27), they are equivalent to the Abelian Higgs model which, in turn, is the reduced form of the Y-M functional for pure gauge fields.*

We continue with the discussion of Witten's treatment of Eq.s(5.17) since we shall need his results later on. First, he selects physically convenient gauge condition via $\partial_\mu A_\mu = 0$. This leads to the choice: $A_\mu = \varepsilon_{\mu\nu}\partial_\nu \psi$ (for some scalar ψ). With such a choice for A_μ the first two of Eq.s(5.17) can be rewritten as

$$(\partial_0 - \partial_0 \psi)\varphi_1 = (\partial_1 - \partial_1 \psi)\varphi_2, \qquad (5.23a)$$

$$(\partial_1 - \partial_1 \psi)\varphi_1 = -(\partial_0 - \partial_0 \psi)\varphi_2. \qquad (5.23b)$$

Let now $\varphi_1 = e^\psi \chi_1$ and $\varphi_2 = e^\psi \chi_2$. Then the above equations are reduced to the Cauchy-Riemann-type equations: $\partial_0 \chi_1 = \partial_1 \chi_2$ and $\partial_1 \chi_1 = \partial_0 \chi_2$. Introduce the function $f = \chi_1 - i\chi_2$. Then, the last of Eq.s(5.17) acquires the form

$$-r^2 \nabla^2 \psi = 1 - ff^* e^{2\psi}. \qquad (5.24)$$

Notice that $-r^2 \nabla^2 = -r^2(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2})$ is the hyperbolic Laplacian [90]. Eq.(5.24) is still gauge invariant in the sense that by changing $f \to fh$ and $\psi \to \psi - \frac{1}{2}\ln(hh^*)$ in this equation we observe that it preserves its original form. This is so because $\nabla^2 \ln(hh^*) = 0$ for any analytic function which does not have zeros. If h does have zeros for $r > 0$, then substitution of $\psi \to \psi - \frac{1}{2}\ln(hh^*)$ into Eq.(5.24) produces isolated singularities at these zeros. After these remarks, Eq.(5.24) can be simplified further. For this purpose, let $\psi = \ln r - \frac{1}{2}\ln(ff^*) + \rho$, provided that $\nabla^2 \ln(ff^*) = 0$ for any analytic function f which does not have zeros[21]. Under such conditions we end up with the Liouville equation

$$\nabla^2 \rho = e^{2\rho}. \qquad (5.25)$$

It is of major importance for what follows.

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[21] In the case if it does, the treatment is also possible as explained by Witten. Following his work, we shall temporarily ignore this option.

5.3.2   The Heisenberg spin chain model

The results of Appendix B imply that the L-L Eq.(5.2) (or their hyperbolic equivalent, Eq.(5.8)) could be interpreted in terms of equations for the Serret-Frenet moving triad. Treatment along these lines suitable for immediate applications is given in papers by Lee and Pashaev [91] and Pashaev [92]. Below we superimpose their results with those of our work, Ref.[84], to achieve our goals.

We begin with definitions. A collection of smooth vector fields $\mathbf{n}_\mu(\mathrm{x,t})$, $\mu = 0 - 2$, forming an orthogonal basis is called the "moving frame". If $x \in \mathcal{S}$ where $\mathcal{S}$ is some two dimensional surface, then let $\mathbf{n}_1(\mathrm{x,t})$ and $\mathbf{n}_2(\mathrm{x,t})$ form basis for the tangent plane to $\mathcal{S}$ $\forall x \in \mathcal{S}$. Then, the Gauss map (that is the map from $\mathcal{S}$ to two dimensional sphere S^2 or pseudosphere $S^{1,1}$) is given by $\mathbf{n}_2(\mathrm{x,t}) \equiv \mathbf{s}$. By design, it should obey Eq.(5.1). This observation provides needed link between the spin and the moving frame vectors. Details are given in [91,92] and Appendix B. It should be clear that since one can draw curves on surfaces both formalisms should involve the same elements. The restriction for the curve to lie at the surface causes additional complications in general but nonessential in the present case.

Next, we introduce the combinations $\mathbf{n}_\pm = \mathbf{n}_0 \pm i\mathbf{n}_1$ possessing the following properties

$$(\mathbf{n}_+, \mathbf{n}_+) = (\mathbf{n}_-, \mathbf{n}_-) = 0 \ , \ (\mathbf{n}_+, \mathbf{n}_-) = 2/\kappa^2, \tag{5.26}$$

where $\kappa^2 = 1$ for S^2 and $\kappa^2 = -1$ for $S^{1,1}$ and $\mathbf{H}^2$. Furthermore, $(..,..)$ defines the scalar product (in Euclidean or pseudo-Euclidean spaces). Also,

$$\mathbf{n}_+ \times \mathbf{s} = i\mathbf{n}_+, \mathbf{n}_- \times \mathbf{s} = -i\mathbf{n}_-, \mathbf{n}_- \times \mathbf{n}_+ = 2i\kappa^2 \mathbf{s}. \tag{5.27}$$

In addition, we shall use the vectors

$$q_\mu = \frac{\kappa^2}{2}(\partial_\mu \mathbf{s}, \mathbf{n}_+) \text{ and } \bar{q}_\mu = \frac{\kappa^2}{2}(\partial_\mu \mathbf{s}, \mathbf{n}_-) \tag{5.28}$$

in terms of which the equations of motion for the moving frame vectors look as follows:

$$D_\mu \mathbf{n}_+ = -2\kappa^2 q_\mu \mathbf{s}, \tag{5.29}$$

$$\partial_\mu \mathbf{s} = q_\mu \mathbf{n}_- + \bar{q}_\mu \mathbf{n}_+, \tag{5.30}$$

with covariant derivative $D_\mu = \partial_\mu - \frac{i}{2}V_\mu$ and $V_\mu = -2\kappa^2(\mathbf{n}_1, \partial_\mu \mathbf{n}_0)$. Consider now Eq.(5.30) for $\mu = 1$. Apply to it the operator ∂_1 and use the equations of motion and the definitions just introduced in order to obtain

$$\partial_1^2 \mathbf{s} = (D_1 q_1)\,\mathbf{n}_- + (\bar{D}_1 \bar{q}_1)\,\mathbf{n}_+ - \frac{4}{\kappa^2}\,|q_1|^2\,\mathbf{s}. \tag{5.31}$$

It can be shown that $q_0 = iD_1 q_1$. In view of this, Eq.(5.30) for $\mu = 0$ acquires the following form:

$$\partial_0 \mathbf{s} = iD_1 q_1 \mathbf{n}_- - i\bar{D}_1 \bar{q}_1 \mathbf{n}_+. \tag{5.32}$$

This equation happens to be of major importance because of the following. Multiply (from the left) Eq.(5.31) by $\mathbf{s} \times$ and use Eq.s(5.27). Then (depending

on signature of κ^2) the obtained result is equivalent to the L-L Eq.(5.2) or its pseudoeuclidean version, Eq.(5.8). Furthermore, for this to happen the fields V_μ and q_μ must be subject to the following constraint equations obtainable directly from Eq.s (5.29)

$$D_\mu q_\nu = D_\nu q_\mu, \tag{5.33a}$$

$$[D_\mu, D_\nu] = -2\kappa^2(\bar{q}_\mu q_\nu - \bar{q}_\nu q_\mu). \tag{5.33b}$$

We are going to demonstrate now that these equations are equivalent to Eq.s(5.17) obtained by Witten.

We begin with the following observation. Let indices μ and ν be respectively 1 and 0. Then, taking into account that $q_0 = iD_1 q_1$ we can rewrite Eq.(5.33b) as

$$F_{10} = B_1 = -2\kappa^2 i(\bar{q}_1 D_1 q_1 - q_1 \bar{D}_1 \bar{q}_1). \tag{5.34}$$

Surely, by symmetry we could use as well: $q_1 = -iD_0 q_0$. This would give us an equation similar to Eq.(5.34). Take now the case $\kappa^2 = -1$ (that is consider $S^{1,1}$) in these equations and compare them with the Ampere's law, Eq.(5.22). We notice that these equations are not the same. However, since the G-L model was originally designed for phenomenological (thermodynamical) description of superconductivity (as explained in detail in our work, Ref.[84]), we know that the underlying equations (obtainable from the G-L functional) contain the London equation which reads[22]

$$\boldsymbol{\nabla} \times \mathbf{B} = C\mathbf{B} \tag{5.35}$$

with C being some constant (determined by physical considerations). Evidently, in view of the London (5.35), Eq.s(5.22) and (5.34) become equivalent. Consider now Eq.(5.33a). To understand better this equation, it is useful to rewrite Eq.(5.18) as follows

$$D_0\phi = -iD_1\phi \text{ or } D_0\phi_1 = D_1\phi_0, \tag{5.36}$$

where $\phi_1 = \phi$ and $\phi_0 = -i\phi$. Take into account now that $\phi = a + ib$ and identify ϕ_1 with q_1 and ϕ_0 with q_0. Then, Eq.(5.33b) acquires the following form $(\kappa^2 = -1)$:

$$(\partial_0 V_1 - \partial_1 V_0) = -i4(\bar{\phi}_0\phi_1 - \phi_0\bar{\phi}_1) = -4(a^2 + b^2). \tag{5.37}$$

Looking at Eq.(5.17c) we can make the following identifications: $V_1 = A_1, V_0 = A_0, \pm 2a = \varphi_1, \pm 2b = \varphi_2$. Then, comparison between Eq.s(5.17c) and (5.37) indicates that we are still missing a factor of r^2 in the l.h.s. and 1 in the r.h.s. Looking at Witten's derivation of the Liouville Eq.(5.25), we realize that these two factors are interdependent. By clever choice of the function ψ they can be made to disappear. This makes physical sense since locally the underlying surface is almost flat. This observation makes Eq.s(5.37) and (5.24) (or 5.17c) equivalent.

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[22]This is not the form of the London equation one can find in textbooks. But in our work, Ref.[84], we demonstrated that Eq.(5.35) is equivalent to the London equation.

Corollary 2. *The L-L and 2 dimensional G-L models are essentially equivalent in the sense just described both in Euclidean and in Minkowski spaces.*

Corollary 3. *The "hyperbolc" L-L Eq.(5.14) or its Euclidean analog should be identified with Floer's Eq.(4.6).*

These results play an important role in the rest of this work and, in particular, in the next subsection.

5.4   The proof (implementation)

5.4.1   General remarks

In Ref.[79], we demonstrated how treatment of combinatorial data associated with real scattering experiments leads to restoration of the underlying quantum mechanical model reproducing the meson spectrum. It was established that the underlying microscopic model is the Richardson-Gaudin (R-G) XXX spin chain model originally designed for description of spectrum of excitations in the Bardeen-Cooper-Schriefer (BCS) model of superconductivity. Subsequently, the same model was used for description of spectra of the atomic nuclei. Since the energy spectrum of the BCS model has the famous gap between the ground and the first excited state, the problem emerges :

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To answer this question affirmatively the "equivalence principle" discovered by L.Witten is very helpful. Using it, we can proceed with quantization of pure Y-M fields by using results by Korotkin and Nicolai, Ref.[31], for gravity. By comparing the main results of our paper, Ref.[79], done for QCD, with those of Ref.[31], done for gravity, we found a complete agreement. In particular, the Knizhnik-Zamolodchikov Eq.s(4.14),(4.15) and the R-G Eq.(4.29) of Ref.[79] coincide respectively with Eq.s(4.27),(4.26) and (4.50) of Ref.[31] even though methods of deriving of these equations are entirely different! Both Ref.s [79] and [31] do not reveal the underlying physics sufficiently deeply though. In the remainder of this section we shall explain why this is indeed so and demonstrate ways this deficiency can be corrected. Experimentally the challenge lies in designing scattering experiments providing clean information about the spectrum of glueballs. Thus far this task was accomplished only in lattice calculations done for unphysically large number of colors, e.g. $N_c \to \infty$.[23]. When it comes to interpreting *real* experiments (always having only three colors to consider[23]), the situation is even less clear, e.g. see Ref.[93]. Hence, the gap problem is full of challenges for both theory and experiment. Fortunately, at least theoretically, the problem does admit physically meaningful solution as we explained already. We continue with ramifications in the next subsection.

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[23]E.g. read Section **6**.

5.4.2 From Landau-Lifshitz to Gross-Pitaevskii equation via Hashimoto map

Since the F-S model is believed to be capable of describing QCD vacua and is also capable of describing knotted/linked structures [17], two questions arise: a) Is this the only model capable of describing QCD vacua? b) To what extent it matters that the F-S model is also capable of describing knots and links? The negative answer to the first question follows from Corollary 3 implying that, in principle, both Euclidean and hyperbolic versions of the L-L equation are capable of describing QCD vacua: different vacua correspond to different steady-state solutions of the L-L equations. The negative answer to the second question can be found in a review, Ref.[85], by Annalisa Calini. From this reference it follows that, besides the F-S model, knotted/linked structures can be also obtained by using standard (that is Euclidean) L-L equation, e.g. see Eq.(B.4) of Appendix B. This fact still does not explain why knots/links are of importance to QCD. We address the above issues in more detail in Section 6. In view of what is said above, wether or not the hyperbolic version of L-L equation is capable of describing knotted structures is not immediately important for us. Far more important is the connection between the hyperbolic L-L and the Ernst equation. Only with this connection it is possible to reproduce results by Korotkin and Nicolai [31].

Eq.(3.19) is just the F-S functional without winding number term. When the commutation relations for su(1,1) introduced in subsection 5.2 are replaced by those for su(2) this leads to the standard L-L equation (instead of Eq.(5.14)). This replacement causes us to abandon the connection with Ernst equation and, ultimately, with the results of Ref.[31]. In such a case the gap problem should be investigated from scratch. In Ref.[25] Faddeev and Niemi indicated that, unless some amendments to the F-S model are made, it is gapless. At the same time from Appendix B it is known that the L-L equation associated with the F-S model can be transformed into the NLSE with help of the Hashimoto map. Recently, Ding [94] and Ding and Inoguchi [95] were able to find analogs of the Hashimoto map for the vortex filaments in hyperbolic, de Sitter and anti de Sitter spaces. It is helpful to describe their findings using terminology familiar from physics literature [96].This leads us to the discussion of properties of the Gross-Pitaevskii equation known in mathematics as the NLSE. In the system of units in which $\hbar = 1$ and $m = 1/2$ this equation can be written as [86]

$$i\psi_t = -\psi_{xx} + 2\kappa \left(|\psi|^2 - c^2 \right) \psi = 0. \qquad (5.38)$$

Zakharov and Shabat [97,98] performed detailed investigation of this equation for both positive and negative values of the coupling constant κ. For $\kappa < 0$ the above equation is used for description of knots/links [85]. The standard Hashimoto map brings the L-L equation associated with the truncated F-S model to the NLSE with $\kappa < 0$ [94, 95]. From the same references it can be found that the Hashimoto-like map brings the (hyperbolic) L-L-like equation to the NLSE for which $\kappa > 0$. Zakharov and Shabat studied in detail differences

in treatments of the NLSE for both negative and positive coupling constants. This difference is caused by difference in underlying physics which in both cases can be explained in terms of the properties of non ideal Bose gas [99,100]. The attentive reader might have noticed at this point that Eq.(5.38) apparently contains no information about the number of particles in such a gas. This parameter, in fact, is hidden in the constant c (the chemical potential) or it can be obtained selfconsistently with help of Eq.(5.38) (from which c is removed in a way described in Appendix B) as explained in Ref.[100]. With this information at our disposal we are ready to make the next step.

5.4.3   From non ideal Bose gas to Richardson-Gaudin equations

Even though statistical mechanics of 1-d interacting Bose gas was considered in detail by Lieb and Linger [101], solid state physics literature is full of refinements of their results up to moment of this writing. These refinements have been inspired by experimental and theoretical advancements in the theory of Bose condensation [96]. Among this literature we selected Ref.s[102,103] as the most relevant to our needs.

Following [102], the Hamiltonian for N interacting bosons moving on the circle of length L is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\check{c} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) \qquad (5.39)$$

with constant $2\check{c}$ coinciding with 2κ in the system of units $\hbar = 1$ and $m = 1/2$. The case $\check{c} > 0$ (repulsive Bose gas) corresponding to the L-L equation in the hyperbolic plane/space happens to be of immediate relevance. *Only for this case it is possible to establish the connection with work by Korotkin and Nicolai* [31]!

We begin by noticing that in the standard BCS theory of superconductivity electrons are paired into singlets (Cooper pairs) with zero centre of mass momentum. The pairing interaction term in this theory accounts only for pairs of attractive electrons with opposite spin and momenta so that the degeneracy for each energy state is a doublet, with level degeneracy $\Omega = 2$ [24]. In the interacting *repulsive* Bose gas model byRichardson [104] to mimic this pairing he coupled two bosons with opposite momenta $\pm k_j$ into one (pseudo) Cooper pair. An assembly of such formed pairs forms *repulsive* Bose gas which in the simplest case is described by the Hamiltonian, Eq.(5.39). *Hence, the fermionic BCS-type model with strong attractive pairing interaction can be mapped into bosonic repulsive model proposed by Richardson.* Although the idea of such mapping looks very convincing, its actual implementation in Ref.[102] has some flaws. Because of this, we shall use results of this reference selectively. For this purpose, fist of all we need to make an explicit connection between the repulsive Bose gas model described by Eq.(5.39) and the model proposed by Richardson. In the weak coupling limit $\check{c}L \ll 1$ the Bethe ansatz equations for the repulsive

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[24]We use here the same notations as in our work, Ref.[**94**].

Bose gas model described by the Hamiltonian, Eq.(5.39), acquire the following form:

$$k_i = \frac{2\pi d_i}{L} + \frac{2\check{c}}{L} \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \frac{1}{k_j - k_i}, i = 1, ..., N. \qquad (5.40)$$

Here $d_i = 0, \pm 1, \pm 2, ...$ denote the excited states for fixed N. To link this result with Richardson's (repulsive boson) model, consider the case of even number of bosons and make $N = 2M$. Next, consider the ground state of this model first. To the first order in $\check{c}$, it is clear that we can write $k_i = \pm\sqrt{E_i}$. Specifically, let $k_{1,2} = \pm\sqrt{E_1}, k_{3,4} = \pm\sqrt{E_3}, ..., k_{2M-1,2M} = \pm\sqrt{E_M}$. Using these results in Eq.(5.40), with the accuracy just stated, the Bethe ansatz equations after some algebra are converted into the following form:

$$\frac{L}{2\check{c}} + \sum_{\substack{j=1 \\ (j\neq i)}}^{\tilde{M}} \frac{2}{E_j - E_i} = \frac{1}{2E_i}, i = 1, ..., M; \ \tilde{M} \leq M. \qquad (5.41)$$

To analyze these equations, we expect that our readers are familiar with works of both Richardson-Sherman, Ref.[105], and Richardson, Ref.[104]. In [105] diagonalization of the pairing force Hamiltonian describing the BCS-type superconductivity was made. Such a Hamiltonian is given by

$$H = \sum_f 2\varepsilon_f \hat{N}_f - g \sideset{}{'}\sum_f \sideset{}{'}\sum_{f'} b_f^\dagger b_{f'}, \qquad (5.42)$$

where $\hat{N}_f = \frac{1}{2}(a_{f+}^\dagger a_{f-} + a_{f-}^\dagger a_{f-}), b_f = a_{f-}a_{f+}$, with $a_{f\sigma}^\dagger$ and $a_{f\sigma}$ being *fermion* creation and annihilation operators obeying usual anticommutation relations $[a_{f\sigma}, a_{f'\sigma'}^\dagger]_+ = \delta_{\sigma\sigma'}\delta_{ff'}$, where $\sigma = \pm$ denotes states conjugate under time reversal. The above Hamiltonian is diagonalized along with the seniority operators (taking care of the number of unpaired fermions at each level f) defined by

$$\hat{\nu}_f = a_{f+}^\dagger a_{f-} - a_{f-}^\dagger a_{f-}. \qquad (5.43)$$

By construction, $[H, \hat{N}_f] = [H, \hat{\nu}_f] = 0$. The classification of the energy levels is done in such a way that the eigenvalues ν_f of the operator $\hat{\nu}_f$ (0 and σ) are appropriate for the case when $g = 0$. This observation allows us to subdivide the Hamiltonian into two parts: H_1, *i.e.* that which does not contain Cooper pairs (for which $\nu_f = \sigma$) and H_2, *i.e.* that which may contain such pairs (for which $\nu_f = 0$). The matrix elements of H_2 are calculated with help of the *bosonic-type* commutation relations

$$[b_{f'}, \hat{N}_{f'}] = \delta_{ff'} b_f \ \text{and} \ [b_f, b_{f'}^\dagger] = \delta_{ff'}(1 - 2\hat{N}_{f'}). \qquad (5.44)$$

These commutators are bosonic but nontraditional. In the traditional case we have $[b_f, b_{f'}^\dagger] = \delta_{ff'}$. We refer our readers to Ref.[105] for details of how this commutator difficulty is resolved. In the light of this resolution, in Ref.[104]

Richardson proposed to deal with the interacting bosons model from the beginning. *Supposedly, such bosonic model can be designed to reproduce results of the fermionic pairing model of* Ref.[105]. An attempt to do just this was made in Ref.[102]. In the repulsive boson model by Richardson the "pairing" Hamiltonian is given by[25]

$$H = \sum_l 2\varepsilon_l \hat{n}_l + \frac{g}{2} {\sum_f}' {\sum_{f'}}' A_f^\dagger A_{f'}. \tag{5.45}$$

in which $\hat{n}_l$ and $A_{f'}$ are bosonic analogs of $\hat{N}_f$ and b_f. It is essential that the sign of the coupling constant g is nonnegative (repulsive bosons). Upon diagonalization, the total energy E is given by

$$E = \sum_{l=1}^n \varepsilon_l \nu_l + \sum_{j=1}^m E_j \tag{5.46}$$

so that summation in the first sum takes place over the unpaired bosons while in the second- over the paired bosons whose energies E_j are determined from the Richardson's equation (Eq.(2.29) of Ref. [104])[26]

$$\frac{1}{2g} + \sum_{l=1}^n \frac{d_l}{2\varepsilon_l - E_k} + \sum_{\substack{i=1 \\ i \neq k}}^m \frac{2}{E_i - E_k} = 0, k = 1, ..., m \tag{5.47}$$

in which n is the total number of single particle (unpaired) levels, m is the total number of pairs, $d_l = \frac{1}{2}(2\nu_l + \Omega_l)$. From [104] it follows that for the bosonic model to mimic the BCS-type pairing model the degeneracy factor $\Omega_l = 1$ and $\nu_l = 0$. It should be noted though that such an identification is not of much help in comparing the repulsive bosonic model with the attractive BCS-type fermionic model (contrary to claims made in Ref.[102]). This can be easily seen by comparison between Eq.(5.47) (that is Eq.(2.29) of Ref.[104]) with such chosen Ω_l and ν_l with Eq.(3.24) of Ref.[105]. By replacing g in Eq.(5.47) by $-g$ we still will not obtain the analog of the key Eq.(3.24) of Ref.[105]! This fact has group-theoretic origin to be explained in the next subsection. In the meantime, Eq.(5.47) still can be used to connect it with Eq.(5.41) originating from different bosonic model described by the Hamiltonian Eq.(5.39). To do so we follow the path different from that suggested in Ref.[102]. Instead, following the original Richardson's paper [104], we let $n = 1$ in Eq.(5.47) then, without loosing generality, we can put $\varepsilon_1 = 0$ so that Eq.(5.47) acquires the following form

$$\frac{1}{E_k} = \frac{1}{2g} + \sum_{\substack{i=1 \\ i \neq k}}^M \frac{2}{E_i - E_k}, \quad k = 1, ..., M. \tag{5.48}$$

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[25]To avoid ambiguities, our coupling constant $\frac{g}{2}$ is chosen exactly the same as in [104].

[26]Since Gaudin's equation is obtained in the limit $|g| \to \infty$. from Eq.(5.47) the spin -like model described by this equation is known as the Richardson-Gaudin (R-G) model.

The rationale for replacing m by M is given on page 1334 of [104]. Evidently, Eq.s (5.41) and (5.48) are identical. This observation allows us to use the Richardson model instead of that described by Eq.(5.39). At first sight such an identification looks a bit artificial. To convince our readers that it does make sense, we would like to use the work by Dhar and Shastry [106,107] on excitation spectrum of the ferromagnetic Heisenberg spin chain. By analogy with Eq.(5.41) these authors derived a similar equation obtained by reducing the Bethe ansatz equations for Heisenberg ferromagnetic chain. It reads[27]

$$\frac{1}{E_l} = \pi d - \frac{d}{n} \sum_{\substack{i=1 \\ i \neq l}} \frac{2}{E_i - E_l}. \tag{5.49}$$

Even though Eq.s(5.48) and (5.49) look almost the same, they are not the same! The crucial difference lies in the signs in front of the second term in the r.h.s. of these equations. Because of this difference Heisenberg's ferromagnetic spin chain model is mapped onto Bose gas model with *attractive* interaction in complete accord with what was said immediately after Eq.(5.38). Regrettably, this result is still not the same as for the BCS-type model investigated in Richardson-Sherman's paper, Ref.[105]. This fact was recognized and discussed in some detail already by Richardson [104]. For completeness, we mention that the problem of BCS-Bose-Einstein condensation (BEC) crossover which follows exactly the qualitative picture just described was made quantitative only very recently in Ref.[108]. Fortunately, it is possible to by-pass this result as explained in the next subsection.

5.4.4 From Richardson-Gaudin to Korotkin-Nicolai equations

In Ref.[109] bosonic and fermionic formalism for pairing models discussed in the previous subsection was developed. This formalism happens to be the most helpful for investigation of the gap problem. Indeed, define three operators $\hat{n}_l = \sum_m a^{\dagger}_{lm} a_{lm}$, $A^{\dagger}_l = (A_l)^{\dagger} = \sum_m a^{\dagger}_{lm} a^{\dagger}_{l\bar{m}}$. They can be used for construction of operators $K^0_l = \frac{1}{2}\hat{n}_l \pm \frac{1}{4}\Omega_l$ and $K^+_l = \frac{1}{2}A^{\dagger}_l = (K^-_l)^{\dagger}$ such that they obey the following commutator algebra

$$[K^0_l, K^+_l] = \delta_{ll} K^+_l, \ \ [K^+_l, K^-_l] = \mp 2\delta_{ll} K^0_l. \tag{5.50}$$

In this algebra as well as in the preceding expressions, the upper sign corresponds to bosons and the lower to fermions. In Ref.[79], we discussed such an algebra for the fermionic case only, e.g. see Eq.s (4.31) of [79]. These results can be extended now for the bosonic case. In fact, such an extension is already developed in Ref.[109]. Unlike [79], where we used $sl(2, \mathbf{C})$ Lie algebra, only its compact version, that is $su(2)$, was used in [109] for representing fermions. For bosonic case the commutation relations, Eq.(5.50), are those for $su(1, 1)$ Lie algebra. Incidentally, in the paper by Korotkin and Nicolai, Ref.[31], exactly the

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[27]The physical meaning of constants entering this equation is not important for us. It is given in Ref.[106]..

same Lie algebra was used. Furthermore, in the same paper it was argued that it is permissible to replace $su(1,1)$ by $sl(2,\mathbf{R})$ Lie algebra while constructing the K-Z-type equations, e.g. read p.428 of this reference. Since in [79] the $sl(2,\mathbf{C})$ Lie algebra was used, that is a complexified version of $sl(2,\mathbf{R})$, this allows us to use many results from our work. Thus, in this subsection we shall discuss only those results of [109] which are absent in our Ref.[79]. In particular, following this reference the set of Gaudin-like commuting Hamiltonians written in terms of operators K_l^0, K_l^+ and K_l^- is given by

$$H_l = K_l^0 + 2g\{\sum_{l'(\neq l)} \frac{X_{ll'}}{2}(K_l^+ K_{l'}^- + K_l^- K_{l'}^+) \mp Y_{ll'} K_l^0 K_{l'}^0\}. \qquad (5.51)$$

Here $X_{ll'} = Y_{ll'} = (\varepsilon_l - \varepsilon_{l'})^{-1}$. For $g \to \infty$ the first term can be ignored and the remainder can be used in the K-Z-type equations. The semiclassical treatment of these equations discussed in detail in [79] is resulting in the following set of Bethe (or R-G) ansatz equations

$$\sum_{l=1}^{n} \frac{d_l}{2\varepsilon_l - E_k} \pm \sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{2}{E_i - E_k} = 0, \ k = 1, ..., m \qquad (5.52)$$

to be compared with Eq.(5.47). Unlike Eq.(5.47), in the present case $d_l = \frac{1}{2}(2\nu_l \pm \Omega_l)$. The bosonic version of Eq.(5.52) corresponding to $su(1,1)$ Lie algebra coincides with Eq.(4.50) of Korotkin and Nicolai paper, Ref.[31], provided that the following identifications are made: $d_l \rightleftarrows s_l$, $2\varepsilon_l \rightleftarrows \gamma_j$. Unlike Ref.[31], where Eq.(5.52) was obtained by standard mathematical protocol, in this work it is obtained based on the underlying physics. Because of this, it is appropriate to extend our physics-stype analysis by considering the case of finite $g's$. Then, Eq.(5.52) should be replaced by

$$\frac{1}{2g} \pm \sum_{l=1}^{n} \frac{d_l}{2\varepsilon_l - E_k} \pm \sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{2}{E_i - E_k} = 0, k = 1, ..., m. \qquad (5.53)$$

In Ref.[31] the gap problem was discussed in detail for the fermionic case when the coupling constant g is negative (BCS pairing Hamiltonian), e.g. see Eq.s (4.43)-(4.45) of Ref.[31]. In the present case we are dealing with the bosonic case for which the coupling constant is positive. Hence the gap problem should be re analyzed. For this purpose, it is convenient to consider both positive and negative coupling constants in parallel for reasons which will become apparent upon reading.

5.4.5   Emergence of the gap and the gap dilemma

Eq.s(5.53) cannot be solved without some physical input.   Initially, such an input was coming from nuclear physics (e.g. read [110-112] for general information on nuclear physics).   Indeed, Richardson's papers [104,105] were written

having applications to nuclear physics in mind. Given this, the question arises about the place of the R-G model among other models describing nuclear spectra and nuclear properties. We need an answer to this question to finish proof of the gap's existence in QCD.

Looking at the Gaudin-like Hamiltonian, Eq.(5.51), and comparing it with the Hamiltonian, Eq.(6), in Ref.[113][28] it is easy to notice that they are almost the same! Because of this, it becomes possible to transfer the methodology of Ref.[113] to the present case. Thus, it makes sense to recall briefly circumstances at which the gap emerges in nuclear physics.

As is well known, the nuclei are made of protons and neutrons. One can talk about the number $\mathcal{N}$ of nucleons, the number Z of protons and the number N of neutrons in a given nucleus. Nuclear and atomic properties happen to be interrelated. For instance, in analogy with atomic physics one can think of some effective nuclear potential in which nucleons can move "independently". This assumption leads to the *shell model* of nuclei. Use of Pauli principle guides fillings of shells the same way as it guides these fillings in atomic physics. This leads to emergence of magic numbers 2, 8, 20, 28, 50, 82 and 126 for either protons or neutrons for the totally filled shells. Accordingly, the most stable are the doubly magic (for both protons and neutrons) nuclei. It is of interest to know what kinds of excitations are possible in such shell models? The simplest of these is when some nucleon is moving from the closed shell to the empty shell thus forming *a hole*. When the number of nucleons increases, the question about the validity of the shell model emerges, again in analogy with atomic physics. As in atomic physics, one can think about the Hartree-Fock (H-F) and other many-body computational schemes, including that developed by Richardson-Sherman and Gaudin. For our purposes, it is sufficient to use only the Tamm-Dankoff (T-D) approximation to the H-F equations described, for example, in Ref.[112]. The essence of this approximation lies in restricting the particle-hole interactions to nucleons lying in the same shell. *The T-D approximation is obtainable from the* is *R-G Eq.s(5.53) when the last term (effectively taking care of Pauli principle) in these equations is dropped.* The T-D approximation was successfully applied for description of the giant nuclear dipole resonance [110-112]. At the classical level the physics of this resonance was explained in the paper by Goldhaber and Teller [114]. The resonance is caused by two nuclear vibrational modes: one, when protons and neutrons move in the opposite directions and another- when they move in the same direction. Upon quantization of such classical model and taking into account the isotopic spin of nucleons, the truncated Eq.s(5.53) are obtained in which both signs for the coupling constant are allowed since the nucleon system is expected to be in two isospin states : $T = 1$ and $T = 0$[29]. Details of these calculations are given in Ref.[112], page 221. Solutions of the T-D equations can be obtained graphically in complete analogy with that described in our work, Ref.[79]. These graphical solutions reflect the particle-hole duality built into the T-D approximation. Because of this duality,

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[28]Published in 1961!

[29]This can be easily understood based on the fact that isospin for both particles and holes is equal to 1/2 [110-112].

the magnitude of the gap in both cases should be the same. To demonstrate this, the *seniority* scheme described in [110-112] is helpful. The seniority operator was defined by Eq.(5.43). It determines the number of unpaired particles in the nuclear system. Since it commutes with the Hamiltonian, the many-body states can be classified with help of its eigenvalues ν_f. Suppose at first that all single particle energies ε_f are the same (that is $\varepsilon_f = \varepsilon$) so that all seniority eigenvalues ν_f are ν. Let then $\mathcal{N}$ be the total number of nucleons. Thus, the state for which $\nu = 0$ contains only pairs, analogously, the state $\nu = 1$ contains just one unpaired nucleon, $\nu = 2$ has 2 unpaired nucleons and $\mathcal{N}$ should be even and so on. So, states $\nu = 0, \nu = 2, \nu = 4, ...$ can exist only in even nuclei. For such nuclei the gap is nonzero. To see this, we follow Refs.[110-112] which we would like now to superimpose with the results of the Richardson-Sherman paper, Ref.[105]. Specifically, on page 231 of this reference one can find the following result for the ground state ($\nu = 0$) energy

$$E_{\nu=0}(N) = 2N\varepsilon - gN(\Omega - N + 1) \qquad (5.54)$$

where N is the number of pairs. To connect this result with that in Refs.[110-112], let $N = \mathcal{N}/2$ and consider the difference

$$\mathcal{E}_{\nu=0}(N) = E_{\nu=0}(\mathcal{N}/2) - \mathcal{N}\varepsilon = -\frac{g}{4}\mathcal{N}(2\Omega - \mathcal{N} + 2). \qquad (5.55)$$

The obtained result coincides with Eq.(11.14) of Ref.[112] as required. To obtain states of seniority $\nu = 2n$ we use Eq.(3.2) of Ref.[105]. It reads

$$E_{\nu=2n}(N) = 2N\varepsilon - g\left(N - n\right)\left(\Omega - N - n + 1\right), \ n = 0, ..., N. \qquad (5.56)$$

Repeating the same steps as in $\nu = 0$ case we obtain,

$$\mathcal{E}_{\nu}(\mathcal{N}) = -\frac{g}{4}\left(\mathcal{N}\text{-}\nu\right)\left(2\Omega - \mathcal{N}\text{-}\nu + 2\right). \qquad (5.57)$$

Finally, consider the difference

$$\mathcal{E}_{\nu}(\mathcal{N}) - \mathcal{E}_{\nu=0}(\mathcal{N}) = \frac{g}{4}\nu(2\Omega - \nu + 2). \qquad (5.58)$$

This result is in accord with Eq.(11.22) of Ref.[112]. Since the obtained difference is $\mathcal{N}$-independent it can be used both ways: a) for calculations in the thermodynamic limit $\mathcal{N} \to \infty$ and b) for making accurate calculations in the opposite limit of very small number of nucleons. In the simplest case we should consider only one shell and the first excited state of seniority 2 for this shell. Initially (the ground state) we have just one pair while finally (the first excited state) we have two independent particles occupying single particle levels.

Looking at Eq.s(5.53) and letting there $m = 1$(one pair) we recognize that the second sum in this set of equations disappears. Thus, by design, we are left with the T-D approximation. Using Eq.(5.58) for $\nu = 2$ we obtain the following value of the gap Δ :

$$\Delta = \mathcal{E}_2(\mathcal{N}) - \mathcal{E}_0(\mathcal{N}) = g\Omega. \qquad (5.59\text{a})$$

Notice, that since Ω is the degeneracy, there could be no more than $\mathcal{N} = \Omega$ particles at the single particle level. Thus, in general we should have $\mathcal{N} \leq \Omega$. Because of the particle-hole duality, it is permissible to look also at the situation for which $\mathcal{N} \geq \Omega$. This is equivalent to changing the sign in front of the coupling constant. Repeating again all steps leads to the final result for the gap

$$\Delta = \mathcal{E}_2(\mathcal{N}) - \mathcal{E}_0(\mathcal{N}) = |g| \, \Omega. \qquad (5.59b)$$

It is demonstrated in Ref.s [110-112] that in the limit $\mathcal{N} \to \infty$, when the continuum approximation (replacing summation by integration) can be used leading to a more familiar BCS-type equation for the gap, the result just obtained survives. Indeed, in Ref.[103] the BCS-type result is obtained in the continuum approximation for the *attractive* Bose gas. In view of the results just obtained, it should be clear that such a result should hold for both attractive and repulsive Bose gases. This conclusion is in accord with accurate recent Bethe ansatz calculations done in Ref.[115] for systems of finite size. Thus, we just arrived at the issue which we shall call *the gap dilemma*. While the results obtained above strongly favor use of the repulsive Bose gas model *(not linked with the F-S model)*, the results obtained in this subsection indicate that, after all, the F-S model *(linked with the attractive Bose gas model)* can also be used for description of the ground and excited states of pure Y-M fields. THE ESSENCE OF THE DILEMMA LIES IN DECIDING WHICH OF THESE RESULTS SHOULD ACTUALLY BE USED.

While the answer is provided in the next section, we are not yet done with the gap discussion. This is so because the seniority model is applicable only to the case when all single-particle levels have the same energy. This is too simplistic. We would like now to discuss more realistic case

Before doing so, few comments are appropriate. In particular, with all successes of nuclear physics models, these models are much less convincing than those in atomic physics. Indeed, all nuclei are made of hadrons which are made of quarks and gluons. Thus the excitations in nuclei are in fact the excitations of quark-gluon plasma. This observation qualitatively explains why the R-G equations work well both in nuclear and particle physics. Some attempts to look at the processes in nuclear physics from the standpoint of hadron physics can be found in Refs.[116,117].

Now we can return to the discussion of the T-D equations. Fortunately, detailed analytical study of these equations was recently made in Ref.[118]. The same authors extended these results to the case of two pairs in [119]. Since the results obtained in [119] are in qualitative agreement with those obtained in Ref.[118], we shall focus attention of our readers only on results of Ref.[118]. Thus, we need to find some kind of analytic solution of the following T-D equation

$$\sum_{i=1}^{L} \frac{\Omega_i}{2\varepsilon_i - E} = \frac{1}{g}. \qquad (5.60)$$

For different $\varepsilon_i' s$ normally it should have L eigenvalues E_μ ($1 \leq \mu \leq L$). Since we are interested in finding the gap, the above equation is written for just one

nucleon pair. Thus the seniority $\nu = 0$. It is of interest to check first what happens when all $\varepsilon'_i s$ coalesce. In such a case we obtain,

$$\frac{\Omega}{2\bar{\varepsilon} - E} = \frac{1}{g}, \tag{5.61}$$

where $\Omega = \sum_i \Omega_i$ and $\varepsilon_i = \bar{\varepsilon} \; \forall i = 1, ..., L$. Eq.(5.61) can be equivalently rewritten as

$$E_0 = 2\bar{\varepsilon} - \Omega g. \tag{5.62}$$

This result for the ground state is in agreement with Eq.(5.54) for $N = 1$. The first excited state is made of one broken pair so that the pairing disappears and the energy $E_{\nu=2} = 2\bar{\varepsilon}$. From here, the value of the gap is obtained as $E_{\nu=2} - E_0 = g\Omega$ in agreement with Eq.(5.59). If now we make all energy levels different, then one can see that solutions to Eq.(5.60) are subdivided into those lying in between the single particle levels (*trapped solutions*) and those which lie outside these levels (*collectivized solutions*). For $|g|$ sufficiently large the solution, Eq.(5.61), is the leading term (in the sense described below) representing the collectivized solution. Since the trapped solutions represent corrections to energies of single particle states, they do not contribute directly to the value of the gap. They do contribute to this value indirectly. Indeed, following Ref.[118] we rewrite Eq.(5.60) as

$$\sum_{i=1}^{L} \frac{\Omega_i}{2\varepsilon_i - E} = \frac{1}{2\bar{\varepsilon} - E} \sum_i \frac{\Omega_i}{1 + 2\frac{\varepsilon_i - \bar{\varepsilon}}{2\bar{\varepsilon} - E}} = \frac{1}{g} \tag{5.63}$$

and expand the denominator of Eq.(5.63) in a power series. As result, the following expansion

$$\frac{E - 2\bar{\varepsilon}}{g\Omega} = -1 - \alpha^2 + \gamma\alpha^3 + O(\alpha^4) \tag{5.64}$$

is obtained in which $\bar{\varepsilon} = \frac{1}{\Omega}\sum_i \Omega_i\varepsilon_i$, $\alpha = \frac{2\sigma}{g\Omega}$, $\sigma = \sqrt{\frac{1}{\Omega}\sum_i \Omega_i(\varepsilon_i - \bar{\varepsilon})^2}$ and γ is related to the higher order moments (details are in Ref.s[118,119]). Using these results, the gap is obtained in the same way as before.

The quality of computations in Ref.[118] was tested for 3-dimensional harmonic oscillator (by adjusting dimensionality of this oscillator it can be thought of as "closed string model" representing both the shell model for atomic nucleus and the gluonic ring for the Y-M fields) for which $\varepsilon_i = (i + 3/2)$ (in the system of units in which $\hbar\omega = 1$) and $\Omega_i = (i + 1)(i + 2)/2$. For this 3-dimensional oscillator corrections to the collectivized energy, Eq.(5.64), become negligible already for $|g| \geq 0.2$, provided that $L \geq 8$. Obtained results allow us to close this section at this point. These results are of no help in solving the gap dilemma though. This task is accomplished in the next section.

6 Resolution of the gap dilemma

6.1 Motivation

In the previous section we provided evidence linking the gap problem for Y-M fields with the problem about the excitation spectrum of the repulsive Bose gas. The gap equation, Eq.(5.59), is also used in nuclear physics where it is known to produce the same value for the gap for both signs of the coupling constant g. Since both options are realizable in Nature in the case of nuclear physics, the question arises about such possibility in the present case. In the case of nuclear physics experimental realization (giant nuclear dipole resonance) of both options for the coupling constant is experimentally testable. Thus, in the present case we have to find some alternative physical evidence. If, indeed, such evidence could be found, this would allow us to bring back into play the well studied F-S model which microscopically is essentially equivalent to the XXX 1d Heisenberg ferromagnet as results of Appendix B and subsections 3.5 and 5.2 indicate. The next subsection supplies us with the alternative physical evidence.

6.2 Some facts about harmonic maps and their uses in general relativity

Suppose we are interested in a map from $m-$dimensional Riemannian manifold $\mathcal{M}$ with coordinates x^a and metric $\gamma_{ab}(\mathbf{x})$ to n-dimensional Riemannian manifold $\mathcal{N}$ with coordinates φ^A and metric $G_{AB}(\varphi)$. A map $\mathcal{M} \to \mathcal{N}$ is called *harmonic* if $\varphi^A\,(x^a)$ satisfies the Euler-Lagrange (E-L) equations originating from minimization of the following Lagrangian

$$\mathcal{L} = \sqrt{\gamma}G_{AB}(\boldsymbol{\varphi})\gamma^{ab}(\mathbf{x})\varphi^A_{,a}\varphi^B_{,b} \tag{6.1}$$

in which $\gamma = \det(\gamma_{ab})$. Since such defined Lagrangian is part of the Lagrangian given by Eq.(3.6), the E-L equations for Eq.(6.1), in fact, coincide with Eq.s(3.10). In the most general form they can be written as [38][30]

$$\varphi^{A;a}_{,a} + \Gamma^A_{BC}\varphi^B_{,a}\varphi^{C,a} = 0. \tag{6.2}$$

In such a form we can look at transformations $\varphi^{A'} = \varphi^{A'}(\varphi^B)$ keeping $\mathcal{L}$ form-invariant. To find such transformations, following Neugebauer and Kramer [38], we introduce the auxiliary Riemannian space defined by the metric

$$dS^2 = G_{AB}(\boldsymbol{\varphi})d\varphi^A d\varphi^B. \tag{6.3}$$

Use of the above metric allows us to investigate the invariance of $\mathcal{L}$ with help of standard methods of Riemannian geometry. In the present case, this means that one should study Killing's equations in spaces with metric G_{AB}. Specifically, let us consider the Lagrangian for source-free Einstein-Maxwell fields admitting at

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[30]We use the 1st edition of Ref.[38] for writing this equation. This means that we have to define Γ^A_{BC} as $\Gamma^A_{BC} = \frac{1}{2}G^{ad}\{\frac{\partial}{\partial\varphi^c}G_{bd} + \frac{\partial}{\partial\varphi^b}G_{cd} - \frac{\partial}{\partial\varphi^d}G_{bc}\}$.

least one non-null Killing vector ξ. To design such a Lagrangian we begin with the Ernst equation, Eq.(2.4), for pure gravity and replace the Ernst potential $\epsilon = -F + i\omega$[31] by two complex potentials $\mathcal{E}$ and Φ. Then, by symmetry, the equations for stationary Einstein-Maxwell fields can be written as follows [38]

$$F\mathcal{E}^{;a}_{,a} + \gamma^{ab}\mathcal{E}_{,a}(\mathcal{E}_{,b} + 2\Phi_{,b}\bar{\Phi}) = 0, F\Phi^{;a}_{,a} + \gamma^{ab}\Phi_{,a}(\mathcal{E}_{,b} + 2\Phi_{,b}\bar{\Phi}) = 0. \qquad (6.4)$$

These equations are obtained by minimization of the Lagrangian

$$\mathcal{L} = \sqrt{\gamma}[\hat{R}_{ab} + 2F^{-1}\gamma^{ab}\Phi_{,a}\Phi_{,b} + \frac{1}{2}F^{-2}\gamma^{ab}(\mathcal{E}_{,a} + 2\bar{\Phi}\Phi_{,a})(\mathcal{E}_{,b} + 2\bar{\Phi}\Phi_{,b})], \quad (6.5)$$

i.e. from equations $\frac{\delta\mathcal{L}}{\delta\gamma^{ab}} = 0, \frac{\delta\mathcal{L}}{\delta\Phi} = 0$ and $\frac{\delta\mathcal{L}}{\delta\mathcal{E}} = 0$.Taking these results into account, the auxiliary metric, Eq.(6.3), can now be written as

$$dS^2 = 2F^{-1}d\Phi d\bar{\Phi} + \frac{1}{2}F^{-2}\left|d\mathcal{E} + 2\bar{\Phi}d\Phi\right|^2. \qquad (6.6)$$

The analysis done by Neugebauer and Kramer [38] shows that there are eight independent Killing vectors leading to the following finite transformations :

$$\begin{array}{ll}
\mathcal{E}' = \alpha\bar{\alpha}\mathcal{E}, & \Phi' = \alpha\Phi; \\
\mathcal{E}' = \mathcal{E} + ib, & \Phi' = \Phi; \\
\mathcal{E}' = \mathcal{E}(1 + ic\mathcal{E})^{-1}, & \Phi' = (1 + ic\mathcal{E})^{-1}; \\
\mathcal{E}' = \mathcal{E} - 2\bar{\beta}\Phi - \beta\bar{\beta}, & \Phi' = \Phi + \beta; \\
\mathcal{E}' = \mathcal{E}(1 - 2\bar{\gamma}\Phi - \gamma\bar{\gamma}\mathcal{E})^{-1}, & \Phi' = (\Phi + \gamma\mathcal{E})(1 - 2\bar{\gamma}\Phi - \gamma\bar{\gamma}\mathcal{E})^{-1}.
\end{array} \qquad (6.7)$$

Complex parameters α, β, γ as well as real parameters b and c are connected with these eight symmetries. Evidently, solutions $\mathcal{E}', \Phi'$ are also solutions of Eq.s(6.4), provided that γ^{ab} stays the same. Therefore if, say, we choose some vacuum solution as a "seed", we would obtain, say, the electrovacuum solution in accord with Appendix A. Incidentally, the electrovacuum solutions obtained by Bonnor (Appendix A) cannot be obtained with help of transformations given by Eq.s(6.7). They are considered separately below. These observations allow us to reduce the Lagrangian $\mathcal{L}$ to the absolute minimum without loss of information. In 1973 Kinnersley [38] found that the group of symmetry transformations for the Einstein-Maxwell equations with non null Killing vector is the group SU(2,1) which has eight independent generators. In view of the above mentioned reduction of $\mathcal{L}$ it is sufficient to replace the metric in Eq.(6.6) by a collection of much simpler metric related to each other by transformations Eq.(6.7). All the possibilities are described in the Table 34.1 of Ref.[38]. For our needs we focus only on three of these (much simpler/reduced) metric listed in this table. These are

$$dS^2 = \frac{2d\xi d\bar{\xi}}{(1 - \xi\bar{\xi})^2}, \mathcal{E} = \frac{1 - \xi}{1 + \xi}, \qquad (6.8)$$

$$dS^2 = \frac{2d\Phi d\bar{\Phi}}{(1 - \Phi\bar{\Phi})^2}, \qquad (6.9)$$

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[31]Recall, that $-F = V$ according to notations introduced in connection with Eq.(2.4).

and

$$dS^2 = \frac{-2d\Phi d\bar{\Phi}}{(1 + \Phi\bar{\Phi})^2}. \tag{6.10}$$

The first and the second of these metric correspond to the vacuum state, respectively, with $\Phi = 0$ and $\mathcal{E} = -1$, of pure gravity associated with the subgroup SU(1,1) of SU(2,1). The third metric, Eq.(6.10), corresponds to a subgroup SU(2). It is related to the electrostatic fields ($\mathcal{E} = 1$) such that the space-time becomes asymptotically flat for $\mathcal{E}\rightarrow 0$.It is important that the metric, Eq.(6.10), is related to the vacuum metric, Eq.s(6.8),(6.9), via transformations either listed in Eq.(6.7) or related to these transformations. In particular, the related transformations can be obtained as follows. Using Ref.[38], it is convenient to make the parameters b and c in Eq.s(6.7) complex and to consider all eight complex parameters as independent of their complex conjugates. Under such conditions the metric given by Eq.(6.10) is related to that given by Eq.(6.8) by the simplest complex transformation: $\Phi' = i\xi$ and $\bar{\Phi}' = i\bar{\xi}$. These transformations indicate that, starting with real vacuum solution for pure gravity as a seed, the above transformations are capable of reproducing some electrovacuum solutions. Additional details are discussed below.

These results can be interpreted as follows. While the Ernst functional, Eq.(3.18), is representing pure axially symmetric gravity, the F-S-type functional, Eq.(3.19), should describe some special case of electrovacuum (Maxwell-Einstein) gravity. In view of results of Appendix C, it is possible to use these transformations in reverse (see below), that is to obtain the results for pure gravity from those for electrovacuum. This peculiar "duality" property of gravitational fields provides physically motivated resolution of the gap dilemma and, in addition, it allows us to obtain many new results.

6.3 Resolution of the gap dilemma and SU(3)×SU(2)×U(1) symmetry of the Standard Model

The original F-S-type model thus far is limited only to SU(2) gauge theory. SU(2) gauge theory is known to be used for description of electroweak interactions where, in fact, one has to use the gauge group SU(2)×SU(1) [19]. The hadron physics (that is QCD) requires us to use the gauge group SU(3). This is caused by the fact that quark model of hadrons uses flavors (e.g. u,d,s,c,b, t) labeling quarks of different masses. Each of these quarks can be in three different colors (r,g,b) standing for "red", "green" and "blue". Presence of different colors leads to fractional charges for quarks. Far from the target the scattering products are always colorless. The gauge group SU(3) is used for description of these colors. Although theoretically the number of colors can be greater than three, this number is strictly three experimentally [19]. The results of this work allow us to reproduce this number of colors. For this purpose we have to be able to provide the answer to the following *fundamental question*:

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Can equivalence between gravity and Y-M fields (for SU(2) gauge group) discovered by Louis Witten be extended to the group SU(3)?

Very fortunately, this can be done! For the sake of space, we shall be brief whenever details can be found in literature, e.g. see Refs.[120-122].

To proceed, first, we have to go back to Eq.s(2.14),(2.15) and to modify these equations in such a way that instead of the Ernst Eq.(2.4) for the vacuum (gravity) field we should be able to obtain Eq.s (6.4) for electrovacuum. In the limit $\Phi = 0$ the obtained set of equations should be reducible to Eq.(2.4). As it was noticed by Gürses and Xanthopoulos [120], in general, this task cannot be accomplished. Indeed, these authors demonstrated that the self-duality Eq.s(2.14) for SU(2) and for SU(3) Lie groups look exactly the same for axially symmetric fields. Nevertheless, in the last case, upon explicit computation instead of the vacuum Ernst Eq.(2.4) one gets an electovacuum equations (e.g. see Eq.s(6.4)) which, following Ernst [43], can be explicitly written as

$$\left(\operatorname{Re}\mathcal{E} + |\Phi|^2\right)\nabla^2\mathcal{E} = (\boldsymbol{\nabla}\mathcal{E} + 2\bar{\Phi}\boldsymbol{\nabla}\Phi)\cdot\boldsymbol{\nabla}\mathcal{E}, \qquad (6.11a)$$

$$\left(\operatorname{Re}\mathcal{E} + |\Phi|^2\right)\nabla^2\Phi = (\boldsymbol{\nabla}\mathcal{E} + 2\bar{\Phi}\boldsymbol{\nabla}\Phi)\cdot\boldsymbol{\nabla}\Phi. \qquad (6.11b)$$

These equations are obtained if, instead of the matrix M given by Eq.(2.15), one uses

$$M = f^{-1}\begin{bmatrix} 1 & \sqrt{2}\Phi & -\frac{i}{2}(\mathcal{E} - \bar{\mathcal{E}} - 2\Phi\bar{\Phi}) \\ \sqrt{2}\bar{\Phi} & -\frac{i}{2}(\mathcal{E} + \bar{\mathcal{E}} - 2\Phi\bar{\Phi}) & -i\sqrt{2}\bar{\Phi}\mathcal{E} \\ \frac{i}{2}(\bar{\mathcal{E}} - \mathcal{E} - 2\Phi\bar{\Phi}) & i\sqrt{2}\bar{\mathcal{E}}\Phi & \mathcal{E}\,\bar{\mathcal{E}} \end{bmatrix} \qquad (6.12)$$

in which, instead of the one complex potential $\epsilon = -F + i\omega$ used for solution of the vacuum Ernst Eq.(2.4), two complex potentials $\mathcal{E}$ and Φ are being used. In this expression the overbars denote the complex conjugation and $f = -\frac{1}{2}(\epsilon + \bar{\epsilon} + 2\Phi\bar{\Phi})$. Since the Einstein-Maxwell Eq.s(6.4) (or (6.11)) are invariant with respect to transformations given by Eq.s(6.7), there should be a matrix A with constant coefficients such that the $M' = AMA^\dagger$ will have primed potentials $\mathcal{E}$ and Φ taken from those listed in the set Eq.(6.7). Authors of [120] found explicit form of such A-matrices. However, when instead of matrix M we substitute the matrix M' into self-duality Eq.s(2.14), the combination $M'^{-1}\partial M'$ looses this information. As result, we are left with the following situation: while on the gravity side the matrix $M' = AMA^\dagger$ does allow us to obtain new and physically meaningful solutions from the old ones, on the Y-M side all this information is lost. Thus, the one-to-one correspondence discovered by L.Witten for SU(2) is *apparently* lost for SU(3). Very fortunately, this happens only apparently! This is so because the Neugebauer- Kramer (N-K) transformations described by Eq.s(6.7) do not exhaust all possible transformations which can be applied to the matrix M, Eq.(6.12). Among those which are not accounted by N-K transformations are those by Bonnor [38,123] whose work is mentioned in Appendix A. These are given by

$$\mathcal{E} = \epsilon\bar{\epsilon}; \Phi = \frac{1}{2}(\epsilon - \bar{\epsilon}) = i\omega, \qquad (6.13)$$

where $\epsilon = -F + i\omega$ is solution of the Ernst Eq.(2.4). In view of the results of Appendix A one can be sure that the potentials $\mathcal{E}$ and Φ satisfy Eq.s(6.11). This means that one can use these (Bonnor's) potentials in the matrix M to reproduce Eq.s(6.11). This time, there is one-to one correspondence between the self-duality Y-M and the Einstein-Maxwell equations. Even though this is true, the question immediately arises about relevance of such solutions to the solution of the gap problem discussed in Section 5. In Section 5 the Ernst Eq.(2.4) was used essentially for this purpose while Eq.s(6.11) are seemingly different from Eq.(2.4). Again, fortunately, the difference is only apparent.

From the definition of Bonnor transformations, Eq.(6.13), it follows that the potential $\mathcal{E}$ is real. Also, from the same definition it follows that $|\Phi|^2 = \omega^2$.Introduce now new potential $Z = \mathcal{E} + \omega^2$. For it, we obtain

$$\boldsymbol{\nabla} Z = \boldsymbol{\nabla}(\mathcal{E} + \omega^2) = \boldsymbol{\nabla}\mathcal{E} + 2\omega\boldsymbol{\nabla}\omega = \boldsymbol{\nabla}\mathcal{E} + 2\bar{\Phi}\boldsymbol{\nabla}\Phi. \qquad (6.14)$$

Using this result, Eq.s(6.11) can be rewritten as follows

$$\left(Z\nabla^2 - \boldsymbol{\nabla} Z \cdot \boldsymbol{\nabla}\right) \begin{pmatrix} \mathcal{E} \\ \omega \end{pmatrix} = 0. \qquad (6.15)$$

Furthermore, consider the related equation

$$\left(Z\nabla^2 - \boldsymbol{\nabla} Z \cdot \boldsymbol{\nabla}\right) \omega^2 = 0. \qquad (6.16)$$

Evidently, if it can be solved, then equation $\left(Z\nabla^2 - \boldsymbol{\nabla} Z \cdot \boldsymbol{\nabla}\right)\omega = 0$ can be solved as well. This being the case, the system of Eq.s(6.15) will be solved if the Ernst-type vacuum equation

$$Z\nabla^2 = \boldsymbol{\nabla} Z \cdot \boldsymbol{\nabla} Z \qquad (6.17)$$

of the same type as Eq.(2.4) is solved. The obtained result is opposite to that derived by Bonnor, described in Appendix A (see also works Hauser and Ernst [124] and by Ivanov [125]). This means that, at least in some cases (having physical significance) the self-dual Y-M fields for both SU(2) and SU(3) gauge groups are obtainable as solutions of the Ernst Eq.(2.4). This means that all results of Section 5 obtained for SU(2) go through for the gauge group SU(3).

With these results at our disposal we would like to discuss their applications to the Standard Model [19, 126]. From Ref.[120] it is known that the matrix $M \in$ SU(3) has subgroups which belong to SU(2). In particular, one of such subgroups is obtained if we let $\Phi = 0$ in Eq.(6.12). Then, in view of Eq.(6.17), it is permissible to replace $\mathcal{E}$ by ϵ of Eq.(2.4). Thus, the obtained matrix M is decomposable as $M = M_1 + M_2$, where the matrix M_1 is given by

$$M_1 = f^{-1} \begin{bmatrix} 1 & 0 & \omega \\ 0 & 0 & 0 \\ \omega & 0 & \epsilon\bar{\epsilon} \end{bmatrix}. \qquad (6.18)$$

in agreement with the matrix M defined by Eq.(2.15) since in this case $f = -\frac{1}{2}(\epsilon + \bar{\epsilon}) = F$. At the same time, the matrix M_2 is given by

$$M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (6.19)$$

Using elementary operations with matrices we can represent matrix M in the form

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 1 \\ a & b & 0 \\ b & c & 0 \end{bmatrix} \qquad (6.20)$$

where $a = 1/F$, $b = \omega/F$ and $c = (F^2 + \omega^2)/F$. Such a form of the matrix $\tilde{M}$ is typical for the semidirect product of groups (when group elements are represented by matrices). In general case one should replace $\tilde{M}$ by

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 1 \\ a & b & \alpha_1 \\ b & c & \alpha_2 \end{bmatrix}$$

Since the 2×2 submatrix belongs to SU(2) (because its determinant is 1) normally describing a rotation in 3d space (in view of SU(2)⇌SO(3) correspondence), the parameters α_1 and α_2 are responsible for translation. In this, more general case, the matrix M describes the Galilean transformations, that is a combination of translations and rotations. If the translational motion is one dimensional it can be compactified to a circle in which case we obtain the centralizer of SU(3) as SU(2)×U(1). At the level of Lie algebra su(3) this result was obtained in Ref.[127], pages 232 and 267. Its physical interpretation discussed in this reference is essentially the same as ours. The obtained centralizer is the symmetry group of the Weinberg-Salam model (part of the standard model describing electroweak interactions).

All these arguments were meant only to demonstrate that the F-S-type model, Eq.(3.19), should be used for description of electroweak interactions. For description of strong interactions, in accord with Ref.[120], we claim that the matrix M given by Eq.(6.12) in which $\mathcal{E}$ and Φ are taken from Bonnor's Eq.s(6.13) is *intrinsically* of SU(3) type. That is, *it cannot be obtained from the matrix M (in which $\Phi = 0$) by applications of the N-K transformations*, i.e. there are no transformations of the type $M'(\Phi) = AM(\Phi = 0)A^\dagger$. Therefore, this type of SU(3) matrix should be associated with QCD part of the SM. Hence we have to use the Ernst functional, Eq.(3.18), instead of the F-S-type, Eq.(3.19). These results provide resolution of the gap dilemma. *Evidently, this resolution is equivalent to the statement that the symmetry group of the SM is* SU(3)×SU(2)×U(1). This result should be taken into account in designing all possible grand unified theories (GUT). In the next subsection we shall discuss the rigidity of this result.

6.3.1 Remarkable rigidity of symmetries of the Standard Model and the extended Ricci flow

In addition to Bonnor's transformations there are many other transformations from vacuum to electrovacuum. In particular, in Appendix A we mentioned transformations discovered by Herlt. By looking at Eq.s(A.5)-(A.7) describing these transformations and comparing them with those by Bonnor, Eq.(6.13), it is an easy exercise to check that all arguments leading from Eq.s(6.11) to (6.17) go through unchanged. By using superposition of N-K transformations and those either by Bonnor or by Herlt it is possible to generate a countable infinity of vacuum-to electrovacuum transformations such that they could be brought back to the vacuum Ernst solution, Eq.(6.17), using results of previous subsection. This property of Einstein and Einstein -Maxwell equations we shall call "rigidity". In view of results of previous subsection, this rigidity explains the remarkable empirical rigidity of symmetries of the SM. Indeed, suppose that the color subgroup SU(3) can be replaced by SU(N), N>3. In such a case it is appropriate again to pose a question : Can self-dual Y-M fields-gravity correspondence discovered by L.Witten for SU(2) be extended for SU(N), N>3? In Ref.[128] Gürses demonstrated that, indeed, this is possible but under non-physical conditions. Indeed, this correspondence requires for SU(n+1) self-dual Y-M fields to be in correspondence with the set of n-1 Einstein-Maxwell fields. Since $n = 1$ and $n = 2$ cases have been already described, we need only to worry about $n > 2$. In such a case we shall have many-to-one correspondence between the replicas of electrovacuum and vacuum Einstein fields which, while permissible mathematically, is not permissible physically since the Bonnor-type transformations require one-to-one correspondence between the vacuum and electrovacuum fields. Herrera-Aguillar and Kechkin, Ref.[129], found a way of transforming the compactified fields of heterotic string (e.g. see Eq.(3.12)) into Einstein–multi-Maxwell fields of exactly the same type as discussed in the paper by Gürses [128]. While in the paper by Gürses these replicas of Maxwell's fields needed to be postulated, in [129] their stringy origin was found explicitly. From here, it follows that results obtained in this subsection make the minimal functional, Eq.(3.8), and the associated with it Perelman-like functional, Eq.(3.13), universal. The universality of the associated with it Ricci flow, Eq.s (3.14), has physical significance to be discussed below.

7 Discussion

7.1 Connections with loop quantum gravity

A large portion of this paper was spent on justification, extension and exploitation of the remarkable correspondence between gravity and self-dual Y-M fields noticed by Louis Witten. Such correspondence is achievable only non perturbatively. In a different form it was emphasized in the paper by Mason and Newman [130] inspired by work by Ashtekar, Jacobson and Smolin [131]. It is not too difficult to notice that, in fact, papers [130,131] are compatible with

Witten's result since reobtaining of Nahm's equations in the context of gravity is the main result of Ref.[131]. In this context the Nahm equations are just equations for moving triad on some 3-manifold. Since the connection of Nam's equations with monopoles can be found in Ref.[68] and with instantons in Ref.[132] the link with Witten's results can be established, in principle. Since the authors of [131] are the main proponents of loop quantum gravity (LQG) such refinements might be helpful for developments in the field of LQG. We shall continue our discussion of LQG in the next subsection.

7.2 Topology changing processes, the extended Ricci flow and the Higgs boson

According to the existing opinion the SM does not account for effects of gravity. At the same time, in the Introduction we mentioned that in recent works by Smolin and collaborators [32-34] it was shown that "topological features of certain quantum gravity theories[32] can be interpreted as particles, matching known fermions and bosons of the first generation in the Standard Model". Similar results were also independently obtained in works by Finkelstein, e.g. see Ref.[133] and references therein. In particular, Finkelstein recognized that all quantum numbers describing basic building blocks(=particles) of the SM can be neatly organized with help of numbers used for description of knots. More precisely, with projections of these knots onto some plane. It happens, that for description of all particles of the electroweak portion of the SM the numbers describing trefoil knot are sufficient. The task of topological/knotty description of the entire SM was accomplished to some extent in Ref.[33]. This reference as well as Ref.s[32-34] in addition are capable of describing particle dynamics/transformations. All these works share one common feature: calculations *do not* require Higgs boson. This fact is consistent with results discussed in subsection 4.3.1.

The question arises: Is this feature a serious deficiency of these topological methods or are these methods so superior to other, that the Higgs boson should be looked upon as an artifact of the previously existing perturbative methods used in SM calculations? To answer this self-imposed question requires several steps.

First, we recall that according to the existing opinion the SM does not account for effects of gravity. In such a case all the above results should have nothing in common with the SM which is not true.

Second, the results obtained in this paper indicate that knots/links/braids mentioned above have not only virtual (combinatorial/topological) but also differential-geometric description (Appendix B). Because of this, topological description should be looked upon as complementary to that obtainable with help of the F-S-type models.

Third, it is known that knot/link- describing Faddeev model can be converted into Skyrme model [134]. It is also known that the Skyrme-type models

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[32]That is LQG.

do not account for quarks explicitly, Ref.[68], page 349. This is not a serious drawback as we shall explain momentarily.

Fourth, much more important for us is the fact that the Skyrme model can be used both in nuclear [135] and high energy [136] physics where it is used for description of both QCD (*nicely describing the entire known hadron spectra*) and electroweak interactions.

To account for quarks one has to go back to the Faddeev-type models capable of describing knots/links and to make a connection between these *physical* knots/links and *topological/combinatorial* knots/ links discussed in Refs[32-34,133]. This is still insufficient! It is insufficient because Floer's Eq.(4.7) connects different vacua each is being described by the zero curvature condition Eq.(4.13). It is always possible to look at such a condition as describing some knot/link differential geometrically. With each knot, say in S^3, some 3-manifold is associated. Furthermore such a manifold should be hyperbolic (subsection 3.6), that is either associated with hyperbolic-type knot/links [20,137] in S^3 or with knots/links "living" in hyperboloid embedded in the Minkowski spacetime. Such a restriction is absent in Ref.s[32-34,133]. At the same time the Y-M functional, Eq.(4.12), is defined for a particular 3-manifold whose construction is quite sophisticated. Eq.(4.7) describes processes of topology change by connecting different vacua. Such changes formally are not compatible with the fact that we are dealing with one and the same 3-manifold M×[0,1]. From the mathematical standpoint [11] no harm is made if one considers just this 3-manifold, e.g. read Ref.[11], page 22, bottom. Since particle dynamics is encoded in dynamics of transformations between knots/links, it causes us to consider transitions between different 3-manifolds. These 3-manifolds should be carefully glued together as described in Ref.[11]. In this picture particle dynamics involving particle scattering/transformation is synonymous with processes involving topology change. These are carried out naturally by instantons. Such processes can be equivalently and more physically described in terms of the properties of the (extended) Ricci flow (subsection 3.4) following ideas of Perelman's proof of the Poincare′ conjecture. Indeed, experimentally there is only finite number of stable particles. Without an exception, the end products of all scattering processes involve only stable particles. This observation matches perfectly with the irreversibility of Ricci flow processes involving changes in topology: from more complex-to less complex 3-manifolds. Such Ricci flow model upon development could provide mathematical justification to otherwise rather vague statements by Finkelstein that "more complicated knots (particles) can therefore dynamically decay to trefoils (stable particles)", Ref.[133], page 10, bottom.

7.3 Elementary particles as black holes

In the paper [138] by Reina and Treves and also in [139] by Ernst it was found that for asymptotically flat Einstein-Maxwell fields generated from the vacuum fields by means of transformations of the type described above, in Section 6, the gyromagnetic factor $g = 2$. For the sake of space, we refer our readers to a recent review by Pfister and King [140] for definitions of g and many historical

facts and developments. In [140] it was noticed that such value of g is typical for most of stable particles of the SM. In view of the quantum gravity-Y-M correspondence promoted in this paper, the interpretation of elementary particles as black holes makes sense, especially in view of the following excerpt from Ref.[38], page 526, "There is one-to-one correspondence between stationary vacuum fields with sources characterized by masses and angular momenta and stationary Einstein-Maxwell fields with purely electromagnetic sources, i.e. charges and currents."

Appendix A

Peculiar interrelationship between gravitational, electromagnetic and other fields

Unification of gravity and electromagnetism was initiated by Nordström in 1913- before general relativity was formulated by Einstein. Almost immediately after Einstein's formulation, Kaluza, in 1921, and Klein, in 1926, proposed unification of electromagnetism and gravity by embedding Einstein's 4-dimensional theory into 5 dimensional space in which 5th dimension is a circle. These results and their generalizations (up to 1987) can be found in the collection of papers compiled by Applequist, Chodos and Freund [141]. Regrettably, this collection does not contain alternative theories of unification. Since such alternative theories are much less known/popular to/with string and gravity theoreticians, here we provide a brief representative sketch of these alternative theories.

The 1st unified Einstein-Maxwell theory in 4-dimenssional space-time was proposed and solved by Rainich in 1925. It was discussed in great detail by Misner and Wheeler [142]. After Rainich there appeared many other works on exact solutions of Einstein-Maxwell fields [38]. The most striking outcome of these, more recent, works is the fact that multitude of exact solutions of the *combined* Einstein-Maxwell equations can be obtained from solutions of the *vacuum* Einstein equations.

In 1961 Bonnor [123]obtained the following remarkable result (e.g. read his Theorem 1). Suppose solutions of the vacuum Einstein equations are known. Using these solutions, it is possible to obtain a certain class of solutions of Einstein-Maxwell equations.

In Section 6 we obtained the reverse result: Einstein's solutions for pure gravity were obtained from solutions of the Einstein-Maxwell equations. Without doing extra work, the electrovacuum solution obtained by Bonnor can be converted into that describing propagation of the combined cylindrical gravitational and electromagnetic waves. With some additional efforts one can use the obtained results as an input for results describing the combined gravitational, electromagnetic and neutrino wave propagation [143-144].

The results by Bonnor comprise only a small portion of results connecting static gravity fields with electromagnetic and neutrino fields. The next example belongs to Herlt [38,145]. It provides a flavor of how this could be achieved.

We begin with Eq.(2.5). When written explicitly, this equation reads

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right)u = 0. \tag{A.1}$$

This type of solution is the result of use of the matrix M, Eq.(2.15), in Eq.(2.14b). Nakamura [146] demonstrated that there is another matrix Q given by

$$Q = \left(\begin{array}{cc} f & f\omega \\ f\omega & f^2\omega^2 - \rho^2 f^{-1} \end{array} \right) \tag{A.2}$$

and the associated with it analog of Eq.(2.14b)

$$\partial_\rho(\rho\partial_\rho Q \cdot Q^{-1}) + \partial_z(\rho\partial_z Q \cdot Q^{-1}) = 0 \tag{A.3}$$

leading to the equation analogous to Eq.(A.1), that is

$$\left(\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \partial_z^2\right)\tilde{u} = 0. \tag{A.4}$$

Nakamura demonstrated that the solution $\tilde{u}$ is obtainable from solution of Eq.(A.1). and vice versa. Thus, instead of the Ernst Eq.(2.4) we can use Eq.(A.4). This fact plays crucial role in Hertl's work. In it, he uses Eq.(A.4) to obtain u in Eq.(A.1) as follows

$$\exp(2u) = \left(\tilde{u}^{-1} + G\right)^2 \tag{A.5}$$

with G given by

$$G = \tilde{u}_{,\rho}[\rho(u_{,\rho}^2 + u_{,z}^2) - \tilde{u}\tilde{u}_{,\rho}]^{-1}. \tag{A.6}$$

These results allow him to introduce a potential χ via

$$\chi = \tilde{u}^{-1} - G. \tag{A.7}$$

Using the original work of Ernst [43] as well as Ref.[38], we find that solution of the static axially symmetric coupled Einstein-Maxwell equations is given in terms of complex potentials ϵ and Φ. In particular, in purely electrostatic case one has $\epsilon = \bar{\epsilon} = e^{2u} - \chi$ and $\Phi = \bar{\Phi} = \chi$ while the magnetostatic case is obtained from the electrostatic by requiring $-\Phi = \bar{\Phi} = \psi$ and $\epsilon = \bar{\epsilon} = e^{2u} - \psi$. In this case ψ is just relabeled χ. Ref.[38] contains many other examples of the coupled Einstein-Maxwell equations obtained from the vacuum solutions of Einstein equations.

The above results should be looked upon from the standpoint of fundamental problem of the energy-momentum conservation in general relativity requiring introduction (in the simplest case) of the Landau-Lifshitz (L-L) energy-momentum pseudotensor. The description of more complicated pseudotensors (incorporating that by L-L) can be found in the monograph by Ortin [147]. To this one should add the problem about the positivity of mass in general relativity. The difficulties with these concepts stem from the very basic observation, lying at the

heart of general relativity, that at any given point of space-time gravity field can be eliminated by moving in the appropriately chosen accelerating frame (the equivalence principle). This fact leaves unexplained the origin of the tidal forces requiring observation of motion of at least two test particles separated by some nonzero distance. The explanation of this phenomenon within general relativity framework is nontrivial.It can be found in [148]. In turn, it leads to speculations about the limiting procedure leading to elimination of gravity at a given point[33]. Apparently, this problem is still not solved rigorously[147]. An outstandig collection of rigorous results on general relativity can be found in the recent monograph by Choquet-Bruhat [149] while [150] discusses peculiar relationship between the Newtonian and Einsteinian gravities at the scale of our Solar system.

Conversely, one can think of other fields at the point/domain where gravity is absent as subtle manifestations of gravity. Interestingly enough, such an idea was originally put forward by Rainich already in 1925 ! Recent status of these ideas is given in paper by Ivanov [125]. From such a standpoint, the functional given by Eq.(3.13) (that is the Perelman-like entropy functional) is sufficient for description of all fields with integer spin. With minor modifications (e.g. involving either the Newman-Penrose formalism [143,144] or supersymmeric formalism used in calculation of Seiberg-Witten invariants [66]), it can be used for description of all known fields in nature.

Appendix B
Some facts about integrable dynamics of knotted vortex filaments

B.1 *Connection with the Landau-Lifshitz equation*

Following Ref.[85], we discuss motion of a vortex filament in the incompressible fluid. Some historical facts relating this problem to string theory are given in our recent work, Ref.[84]. Let $\mathbf{u}$ be a velocity field in the fluid such that div$\mathbf{u} = 0$. Therefore, we can write $\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{A}$. Next, we define the vorticity $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$ so that eventually,

$$u = -\frac{1}{4\pi} \int d^3x \frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{w}(\mathbf{x}')}{\|\mathbf{x} - \mathbf{x}'\|^3}. \qquad (B.1a)$$

This expression can be simplified by assuming that there is a *line* vortex which is modelled by a tube with a cross-sectional area dA and such that the vorticity $\mathbf{w}$ is everywhere tangent to the line vortex and has a constant magnitude w. Let then $\Gamma = \int w dA$ so that

$$u = -\frac{\Gamma}{4\pi} \oint \frac{(\mathbf{x} - \mathbf{x}') \times d\boldsymbol{\gamma}}{\|\mathbf{x} - \mathbf{x}'\|^3} \qquad (B.1b)$$

with $d\boldsymbol{\gamma}$ being an infinitesimal line segment along the vortex. Such a model of a vortex resembles very much model used for description of dynamics of ring polymers [84]. Because of this, it is convenient to make the following

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[33]The abundance of available energy-momentum pseudotensors is result of these speculations.

identification : $\mathbf{u}(\boldsymbol{\gamma}(s,t)) = \frac{\partial \boldsymbol{\gamma}}{\partial t}(s,t)$, with s being a position along the vortex contour and t-time. This allows us to write

$$\frac{\partial \boldsymbol{\gamma}}{\partial t}(s',t) = -\frac{\Gamma}{4\pi} \oint \frac{(\boldsymbol{\gamma}(s',t) - \boldsymbol{\gamma}(s,t))}{\|\boldsymbol{\gamma}(s',t) - \boldsymbol{\gamma}(s,t)\|^3} \times \frac{\partial \boldsymbol{\gamma}}{\partial s} ds \qquad (\text{B.1c})$$

and to make a Taylor series expansion in order to rewrite Eq.(B1c) as

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} = \frac{\Gamma}{4\pi} [\frac{\partial \boldsymbol{\gamma}}{\partial s'} \times \frac{\partial^2 \boldsymbol{\gamma}}{\partial s'^2} \int \frac{ds}{|s - s'|} + ...]. \qquad (\text{B.1d})$$

In this expression only the leading order result is written explicitly. By introducing a cut off ε such that $|s - s'| \geq \varepsilon$ and by rescaling time: $t \to \frac{\Gamma}{4\pi} t \ln(\varepsilon^{-1})$ one finally arrives at the basic vortex filament equation

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} = \frac{\partial \boldsymbol{\gamma}}{\partial s'} \times \frac{\partial^2 \boldsymbol{\gamma}}{\partial s'^2}. \qquad (\text{B.2})$$

Introduce now the Serret-Frenet frame made of vectors $\mathbf{B}, \mathbf{T}$ and $\mathbf{N}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}, \check{\kappa}\mathbf{N} = \frac{d\mathbf{T}}{ds}, \mathbf{T} = \frac{\partial \boldsymbol{\gamma}}{\partial s}$, where $\check{\kappa}$ is a curvature of $\boldsymbol{\gamma}$. Then, Eq.(B.2) can be equivalently rewritten as

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} = \check{\kappa}\mathbf{B} \qquad (\text{B.3})$$

or, as

$$\frac{\partial \mathbf{T}}{\partial t} = \mathbf{T} \times \mathbf{T}_{xx}. \qquad (\text{B.4})$$

In the last equation the replacement $s \leftrightharpoons x$ was made so that the obtained equation coincides with the Landau-Lifshitz (L-L) equation describing dynamics of 1d Heisenberg ferromagnets [86].

B.2 *Hashimoto map and the Gross-Pitaevskii equation*

Hashimoto [85] found ingenious way to transform the L-L equation into the nonlinear Scrödinger equation (NLSE) which is also widely known in condensed matter physics literature as the Gross-Pitaevskii (G-P) equation [96]. Because of its is uses in nonlinear optics and in condensed matter physics for description of the Bose-Einstein condensation (BEC) theory of this equation is well developed. Some facts from this theory are discussed in the main text. Here we provide a sketch of how Hashimoto arrived at his result.

Let $\mathbf{T}, \mathbf{U}$ and $\mathbf{V}$ be another triad such that

$$U = \cos(\int^x \tau ds)\mathbf{N} - \sin(\int^x \tau ds)\mathbf{B}, \ \mathbf{V} = \sin(\int^x \tau ds)\mathbf{N} + \cos(\int^x \tau ds)\mathbf{B} \quad (\text{B.5})$$

in which τ is the torsion of the curve. Introduce new curvatures κ_1 and κ_2 in such a way that

$$\kappa_1 = \check{\kappa} \cos(\int^x \tau ds) \text{ and } \kappa_2 = \check{\kappa} \sin(\int^x \tau ds),$$

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then, it can be shown that

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} = -\kappa_2 \mathbf{U} + \kappa_1 \mathbf{V} \tag{B.6a}$$

and

$$\frac{\partial^2 \boldsymbol{\gamma}}{\partial x^2} = \kappa_1 \mathbf{U} + \kappa_2 \mathbf{V}. \tag{B.6b}$$

Using these equations and taking into account that $\mathbf{U}_t \mathbf{V} = -\mathbf{U}\mathbf{V}_t$ after some algebra one obtains the following equation

$$i\psi_t + \psi_{xx} + [\frac{1}{2}|\psi|^2 - A(t)]\psi = 0 \tag{B.7}$$

in which $\psi = \kappa_1 + i\kappa_2$ and $A(t)$ is some arbitrary x-independent function. By replacing ψ with $\psi \exp(-i \int^t dt' A(t'))$ in this equation we arrive at the canonical form of the NLSE which is also known as focussing cubic NLSE.

$$i\psi_t + \psi_{xx} + \frac{1}{2}|\psi|^2 \psi = 0 \tag{B.8}$$

It can be shown that its solution allows us to restore the shape of the curve/filament $\boldsymbol{\gamma}(s,t)$. The G-P equation can be identified with Eq.(B.7) if we make $A(t)$ time-independent. In its canonical form it is written as (in the system of units in which $\hbar = 1$, $m = 1/2$) [86]

$$i\psi_t = -\psi_{xx} + 2\kappa \left(|\psi|^2 - c^2\right)\psi = 0. \tag{B.9}$$

In general, the sign of the coupling constant κ can be both positive and negative. In view of Eq.(B.8), when motion of the vortex filament takes place in Euclidean space, the sign of κ is negative. This is important if one is interested in dynamic of *knotted* vortex filaments [85]. For purposes of this work it is also of interest to study motion of vortex filaments in the Minkowski and related (hyperbolic, de Sitter) spaces. This should be done with some caution since the transition from Eq.(B.1a) to (B.2) is specific for Euclidean space. Thus, study can be made at the level of Eq.s (B.3) and (B.4). Fortunately, such study was performed quite recently [94,95]. The summary of results obtained in these papers can be made with help of the following definitions. Introduce a vector $n = \{n_1, n_2, n_3\}$ so that the unit sphere S^2 is defined by

$$S^2 : n_1^2 + n_2^2 + n_3^2 = 1. \tag{B.10}$$

Respectively, the de Sitter space $S^{1,1}$ (or unit pseudo sphere in $\mathbf{R}^{2,1}$) is defined by

$$S^{1,1} : n_1^2 + n_2^2 - n_3^2 = 1, \tag{B.11}$$

while the hyperbolic space $\mathbf{H}^2$(or hyperboloid embedded in $\mathbf{R}^{2,1}$) is defined by

$$\mathbf{H}^2 : n_1^2 + n_2^2 - n_3^2 = -1, n_3 > 0. \tag{B.12}$$

Using these definitions, it was proven in [94,95] that: a) for both de Sitter and $\mathbf{H}^2$ spaces there are analogs of the L-L equation (e.g. those discussed in the main text, in subsection 5.2.); b) the Hasimoto map can be extended for these spaces so that the respective L-L equations are transformed into the same NLSE (or G-P) equation in which κ is *positive*.

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